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ScienceDirect
Probabilistic Engineering Mechanics 23 (2008) 254–266
PROBABILISTIC ENGINEERING MECHANICS
www.elsevier.com/locate/probengmech

Joint distribution of peaks and valleys in a stochastic process

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Received 26 November 2007; accepted 10 December 2007
Available online 23 December 2007

Abstract

General expressions and numerical results are presented pertaining to the occurrence of two local extrema of a stochastic process at prescribed time values. The extrema may be either peaks or valleys and the process may be either stationary or nonstationary. General formulas are presented for the rates of occurrence, the joint and conditional probability distributions, and the moments of the extreme values. These formulas are relatively simple multiple-integral expressions, but the integrands involve the joint probability density function for six random variables. The procedures are then applied for the special case of a stationary mean-zero Gaussian process for which the calculations are greatly simplified. Numerical results for three different spectral density functions demonstrate that conditioning on either only the existence or both the existence and the value of one peak can have a very significant effect on both the rate of occurrence and the probability distribution of a second peak.

Keywords: Conditional distribution; Extrema; Gaussian; Joint distribution; Occurrence rates; Peaks; Valleys

1. Introduction

Information about the values of the local extrema of a stochastic process is valuable in various models for estimating failure probabilities in mechanical or structural systems. There is the obvious situation regarding the occurrence of a critical level of stress or displacement by the dynamic response of a system, although this usually involves analysis of a global extreme value within a given period of time, rather than a strictly local maximum. Prediction of fatigue life, on the other hand, commonly involves estimation of the magnitude of a stress range from a local minimum (valley) to a local maximum (peak), or vice versa. Studying such a stress range, of course, requires information about the joint distribution of the valley and peak that define its end points. It should be noted that this study of local extrema at given time values is distinctly different from the study of the largest extrema within a given time interval. The latter problem has received much study, including the joint distribution of several of the largest extrema within the interval (e.g., [1]). This is in contrast to the problem studied here, which seems to have received little attention.

Rice [2] gave the probability distribution of a single peak or valley of a stochastic process, although there is sometimes some disagreement as to whether these formulas are exact or approximate. In the formulation used here these formulas are considered to give an exact description of conditional probability distributions [3]. The present work then extends this approach to derive conditional joint distributions for two peaks, two valleys, or a valley and a peak. From these results it is possible to derive conditional distributions (or conditional moments) of one extremum given information about another extremum. Results are presented both for the situation where only the existence of the second extremum is given, and for the case when both the existence and the value of the other extremum are known. Similarly, the usual form of Rice's formula for the rate of occurrence of extrema is extended to give a joint occurrence rate and a conditional rate given the existence of another extremum.

The current work focuses on the derivation of the joint and conditional distributions for a quite general stochastic process, and the specialization and simplification for the special case of a mean-zero, stationary Gaussian process. None of the results require the imposition of any Markov property, but they do require the existence of finite moment properties of the process and its first two derivatives. For the Gaussian situation, this

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0266-8920/$ - see front matter
doi:10.1016/j.probengmech.2007.12.008

requires that the autocorrelation function of the process be four times differentiable.

Numerical results are presented for the extrema of several example autocovariance functions. Plots are given for conditional rates of occurrence, joint probability density, conditional probability density, conditional moments, and the correlation of two peaks. These results demonstrate not only the general behavior of the joint and conditional rates of occurrence and distributions, but also the very significant effect of the behavior of the autocovariance function of the process for very small values of the time parameter, which depends on the high-frequency "tail" of the corresponding autospectral density function.

2. General formulation

Let $\{X(t)\}$ be a stochastic process with continuous and smooth time histories, and let $P_1 = X(t_1)$ denote a peak of the process at time t_1. It must be recognized, though, that this peak occurs only in some of the infinite ensembles of possible time histories. The probability of the occurrence of peak P_1 in a time interval $[t_1, t_1 + \Delta\tau_1]$ is

$$P(P_1 \text{ exists}) = \nu_P(t_1)\Delta\tau_1 \tag{1}$$

in which $\nu_P(t_1)$ denotes the expected rate of occurrence of peaks, and is given by [2]

$$\nu_P(t_1) \equiv \int_{-\infty}^{0} w_1 p_{\dot{X}(t_1)\ddot{X}(t_1)}(0, w_1)\mathrm{d}w_1. \tag{2}$$

The cumulative distribution of the peak is found by investigating the probability of the occurrence of a peak that is both within the given time interval and below a specified value u_1:

$$P(P_1 \le u_1 \text{ exists}) = \Delta\tau_1 \int_{-\infty}^{u_1}\int_{-\infty}^{0} w_1 p_{X(t_1)\dot{X}(t_1)\ddot{X}(t_1)}(r, 0, w_1)\mathrm{d}w_1\mathrm{d}r. \tag{3}$$

Thus, the conditional probability of $P_1 \le u_1$ given that P_1 exists is (3) divided by (1) and the derivative of this ratio with respect to u_1 gives the corresponding conditional probability density function for P_1 as

$$p_{P_1}(u_1|P_1 \text{ exists}) \equiv \frac{\mathrm{d}}{\mathrm{d}u_1}\frac{P(P_1 \le u_1 \text{ exists})}{P(P_1 \text{ exists})} = \frac{\int_{-\infty}^{0} w_1 p_{X(t_1)\dot{X}(t_1)\ddot{X}(t_1)}(u_1, 0, w_1)\mathrm{d}w_1}{\nu_P(t_1)}. \tag{4}$$

These results can be directly extended to describe the behavior of two peaks. In particular, the joint probability of the occurrence of peaks $P_1 = X(t_1)$ and $P_2 = X(t_2)$ in time intervals $[t_1, t_1 + \Delta\tau_1][t_2, t_2 + \Delta\tau_2]$ can be written as

$$P(P_1 \text{ and } P_2 \text{ exist}) = \nu_{PP}(t_1, t_2)\Delta\tau_1\Delta\tau_2 \tag{5}$$

in which the dual occurrence rate for peaks at the two specified times is

$$\nu_{PP}(t_1, t_2) \equiv \int_{-\infty}^{0}\int_{-\infty}^{0} w_1 w_2 p_{\dot{X}(t_1)\ddot{X}(t_1)\dot{X}(t_2)\ddot{X}(t_2)} \times (0, w_1, 0, w_2)\mathrm{d}w_1\mathrm{d}w_2. \tag{6}$$

The conditional probability of P_1 occurring in $[t_1, t_1 + \Delta\tau_1]$, given the existence of P_2 in $[t_2, t_2 + \Delta\tau_2]$ is the joint probability in (5) divided by the probability of P_2 occurring in $[t_2, t_2 + \Delta\tau_2]$. This latter probability is the same as (1), except that t_1 and Δt_1 are replaced by t_2 and Δt_2. Dividing the resulting ratio by $\Delta\tau_2$ then gives the conditional rate of occurrence of peaks at time t_1 given the occurrence of a peak at time t_2:

$$\nu_P(t_1|P_2 \text{ exists}) = \nu_{PP}(t_1, t_2)/\nu_P(t_2). \tag{7}$$

Similarly, one can extend (3) to give a joint cumulative distribution term of the form $P(P_1 \le u_1 \text{ and } P_2 \le u_2 \text{ exist})$, divide by the probability of occurrence in (5), and take the mixed partial derivative with respect to u_1 and u_2 to obtain the joint probability density function of the peaks, given that both occur:

$$p_{P_1P_2}(u_1, u_2|P_1 \text{ and } P_2 \text{ exist}) \equiv \frac{\partial^2}{\partial u_1 \partial u_2}\frac{P(P_1 \le u_1 \text{ and } P_2 \le u_2 \text{ exist})}{P(P_1 \text{ and } P_1 \text{ exist})} = \frac{f_{PP}(u_1, u_2)}{\nu_{PP}(t_1, t_2)} \tag{8}$$

in which

$$f_{PP}(u_1, u_2) \equiv \int_{-\infty}^{0}\int_{-\infty}^{0} w_1 w_2 p_{\mathbf{X}}(u_1, 0, w_1, u_2, 0, w_2) \times \mathrm{d}w_1\mathrm{d}w_2 \tag{9}$$

with the vector $\mathbf{X}$ defined as $\mathbf{X} \equiv \{X(t_1), \dot{X}(t_1), \ddot{X}(t_1), X(t_2), \dot{X}(t_2), \ddot{X}(t_2)\}^{\mathrm{T}}$.

At this point one can observe the importance of being explicit about the conditioning of the various quantities. In common practice the result in (4) is simply designated as the distribution of a single peak, without specification that it has conditioning. If one were to follow this same approach for the joint distribution, then (8) would be considered as the joint distribution of two peaks at the specified times. This would suggest that (4) is the marginal distribution corresponding to the integration of (8) with respect to one of its arguments. This is most assuredly not true, though. In fact, integrating (8) over all possible values of u_2 gives a conditional probability density function for P_1 given that P_1 and P_2 both occur:

$$p_{P_1}(u_1|P_1 \text{ and } P_2 \text{ exist}) = g_{PP}(u_1)/\nu_{PP}(t_1, t_2) \tag{10}$$

in which

$$g_{PP}(u_1) \equiv \int_{-\infty}^{0}\int_{-\infty}^{0} w_1 w_2 \times \left(\int_{-\infty}^{\infty} p_{\mathbf{X}}(u_1, 0, w_1, u_2, 0, w_2)\mathrm{d}u_2\right)\mathrm{d}w_1\mathrm{d}w_2 \tag{11}$$

and this is quite different from (4), in which only one peak is known to exist. Integrating with respect to u_1 instead of u_2 in (11) gives a corresponding term that will be denoted as $g_{PP}(u_2)$. This can then be used to write the conditional probability density function for the second peak as $p_{P_2}(u_2|P_1 \text{ and } P_2 \text{ exist}) = g_{PP}(u_2)/\nu_{PP}(t_1, t_2)$.

An additional probability density function of interest for the random variable P_1 is the one conditioned on the existence of