Mechanics of Materials 77 (2014) 86–97

Contents lists available at ScienceDirect

Mechanics of Materials

journal homepage: www.elsevier.com/locate/mechmat

Inclusion problem of a two-dimensional finite domain: The shape effect of matrix

Chunlin Pan, Qiang Yu *

Department of Civil and Environmental Engineering, University of Pittsburgh, United States

A R T I C L E I N F O

Article history:
Received 31 December 2013
Received in revised form 16 April 2014
Available online 11 July 2014

Keywords:
Inclusion problem
Shape effect
Complex potential method
Finite domain
Conformal mapping
Cauchy integral

A B S T R A C T

With the advance in composite mechanics and micromechanics, there are increasing demands for analytical solutions of inclusion problems in a bounded domain. To echo this need, this study is focused on establishing explicit expressions of elastic fields for a 2D elastic domain containing a circular inclusion at center. Unlike the configuration in the classical Eshelby formulation, the elastic domain in this study is bounded and has shapes other than a circle. To circumvent the mathematical difficulty in solving Green's function in a finite domain, an approach powered by complex potential method, which has been successfully employed to formulate the elastic fields for inclusion problems where matrix is unbounded or bounded by a circle, is extended to finite domains displaying complicated shapes, particularly, a Pascal's limaçon and a curved square (an approximation of perfect square) in this study. In order to take advantage of the mathematical simplicity inherent in expressing a circular geometry, conformal mapping is used to transform the complex geometry of the finite domain of interest to a unit circle. The governing complex potentials, which capture the discontinuity on the inclusion–matrix interface due to the uniform eigenstrain within the inclusion, are formulated with the aid of Cauchy integral and then explicitly identified by satisfying the prescribed boundary conditions. In this study, the displacement fields for finite domains bounded by a Pascal's limaçon and a curved square are obtained based on Dirichlet (displacement) boundary conditions imposed by the far field strain. In addition to asymptotical behaviors, firm agreement is also achieved when the analytical solutions based on complex potentials are compared with the FEM results. Furthermore, inverse of the conformal mapping is discussed here in order to get the explicit expression for elastic fields.

1. Introduction

Due to their significant importance in composite mechanics and micromechanics, inclusion problems in finite domains are attracting increasing attentions in a rich variety of engineering applications, e.g., particle-reinforced composites, cementitious materials, quantum dots, to name a few. The key of the problem is to determine the elastic fields in a bounded domain, which are disturbed by an inclusion embedded at center. Its fundamental difference from the classical Eshelby's formulation is that the domain is now bounded and has a specific shape, which presents a significant effect on the disturbed elastic fields.

Eshelby's pioneer work (Eshelby, 1957, 1959) led to the analytical solutions of the elastic fields for an infinite domain containing an ellipsoidal inclusion. The strains in the inclusion and matrix can be elegantly characterized by the interior and exterior tensors respectively, which

* Corresponding author.
E-mail addresses: chp66@pitt.edu (C. Pan), qiy15@pitt.edu (Q. Yu).

http://dx.doi.org/10.1016/j.mechmat.2014.07.003
0167-6636/

are later named after him. Serving as the cornerstone, the classical Eshelby's tensors are utilized for important homogenization schemes such as Mori–Tanaka homogenization (Mori and Tanaka, 1973; Weng, 1990) and Self-consistent homogenization (Hill, 1965; Huang et al., 1994), both of which are widely used for approximating the average properties and responses of composite materials. Following Eshelby's breakthrough, significant progress has been made to extend the scope of the inclusion problems to take into account the additional interfacial phase (Luo and Weng, 1987, 1989), inclusion shape (Rodin, 1996; Nozaki and Taya, 1997, 2001), orthotropic matrix (Ru, 2003), and matrix-inclusion bond slip (Mura et al., 1985).

However, such progress made in the regime of infinite domain cannot realistically represent a wide spectrum of problems where the matrix is bounded and the volume fraction of the inclusion cannot be ignored. The initial attempts to bridge this gap (Kinoshita and Mura, 1984; Kirchner and Ni, 1993) were not successful because of the mathematic difficulty imposed by seeking for Green's function in a finite domain. Later, using stress function, Luo and Weng (1987, 1989) formulated both the 2D and 3D elastic fields for a 3-phase inhomogeneous inclusion problem with the third phase (i.e., the exterior matrix) being unbounded. This only implicitly provides analytical solutions for a finite domain of 2 phases forming a concentric circular configuration in 2 limiting cases: (1) free boundary if the third phase is infinitely soft and (2) fixed boundary if the third phase is perfectly rigid. Other than that, no general solutions were available for finite domains until recent progress made by Li et al. (2005), Wang et al. (2005) and Li et al. (2007a,b), who took advantage of the geometric characteristics of a concentric circular configuration to postulate that the circumference basis of the Eshelby's tensor of an isotropic finite domain is similar to that of the Eshelby's tensor in an isotropic infinite domain. If this holds, which does for a concentric circular configuration, it means the obstacle of solving Green's function in a finite domain can be bypassed. Using Somigliana's identity and Green's function in infinite domain, general solutions of the exterior and interior tensors were presented for finite domains showing a concentric circular configuration. Subsequently, Gao and Ma (2010) and Ma and Gao (2011) gave the similar solutions based on strain gradient elasticity theory.

However, all these solutions are only effective for representative volume elements (RVEs) of a concentric circular configuration. For matrices of shapes other than a circle, the solutions are not applicable because the validity of the key postulate disappears. To overcome this problem, a versatile approach, whose generality does not rely on the geometric properties of the matrix shape, is needed. Due to its rich theoretical foundation in 2D algebraic problems, effort has been made to use complex potential method to tackle the boundary value problem induced by the bounded matrix.

Initially developed by Kolosov (1909), complex potential method has been widely used to formulate analytical expressions for problems in plane theory of elasticity. Creatively, Sherman (1940) and Muskhelishvili (1953) proposed that one could construct the complex potential functions based on the eigenstrain-induced gap on the inclusion-matrix interface so as to convert the inclusion problem in an infinite domain to the Riemann–Hilbert problem, which can be solved based on the evaluation of certain Cauchy integrals. They further conjectured that via intentional selection of two additional holomorphic functions in the formulation, this approach could be extended to finite domains. However, the attempts (Jaswon and Bhargava, 1961; List and Silberstein, 1966; Sendeckyj, 1970) to practice this conjecture in finite domains did not result in any analytical solutions until recently. Proposing a superposition framework, which is essentially equivalent to customizing the additional holomorphic functions conjectured by Sherman (1940) and Muskhelishvili (1953), Zou et al. (2012) obtained for the first time the general analytical expressions for 2D inclusion problems in a finite domain. However, in their work, mathematical formulation is focused on circular RVEs and the approaches to tackle the shape effect of matrix are not documented.

In many engineering applications, the shape of the RVE cannot be circular because the composite solids cannot be discretized by circular RVEs. Instead, squared, hexagonal and other complicatedly shaped RVEs have to be used. Therefore, the mathematical formulations presented by Sherman (1940), Muskhelishvili (1953) and Zou et al. (2012) have to be extended to take into account the shape effect of the RVE. A great advantage of complex potential method is that the shape effect on the complex potentials can be handled by conformal mapping (Sendeckyj, 1970; Jasiuk et al., 1992). Thus, solutions on a simple geometry like a unit circle, which brings simplicity in mathematical formulation, can be utilized for complex shapes. As documented by Sherman (1940) and Muskhelishvili (1953), the complex potentials for inclusion problems can be expressed by power series, which make it convenient to employ the injective conformal mapping in the form of polynomials to take the shape effect into account.

The objective of this study is to extend the solutions based on complex potential approach, especially those given by Zou et al. (2012) on a circular RVE, to inclusion problems in finite domains of shapes other than a circle. The focus is placed on providing a systematic approach built upon complex potential formulation coupled with conformal mapping transformation of different shapes. The paper is presented as follows. Following the description of the bounded RVEs of complex shapes and corresponding boundary conditions, the key aspects of the conformal mapping to transform the complicatedly shaped matrices to a unit circle are introduced. Then, the general formulations for complex potentials governing the finite domains and their expressions in the mapped domain are delineated. Subsequently, the explicit solutions for the elastic fields are obtained for bounded domains showing shapes of a Pascal's limaçon which can easily recover the concentric circular configuration and of a curved square which gives realistic approximation of a perfect square. In addition to the comparison with classical Eshelby's solution and theoretical solutions for concentric circular RVEs, the solutions are compared with FEM simulations as well. At the end, discussions