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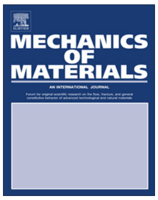

# An elastic–plastic analysis of spherical indentation: Constitutive equations for single-indentation unloading and development of plasticity due to repeated indentation

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The unloading response of elastic–plastic materials indented by a rigid spherical indenter and the development of plasticity due to repeated indentation are examined in the light of finite element simulations performed for a wide range of effective elastic modulus-to-yield strength ratio $E^*/Y$ and strain hardening exponent. Equations of the dimensionless residual indentation depth and plastic work vs maximum indentation depth are extracted from finite element solutions. A relation of the residual indentation depth is derived for isotropic strain hardening materials using the concept of effective strain. Constitutive contact equations are given for elastic–perfectly plastic and isotropic strain hardening materials subjected to indentation unloading. The evolution of plasticity is tracked in four consecutive indentation cycles. For elastic–perfectly plastic materials with high $E^*/Y$, only the first indentation cycle is inelastic, implying elastic shakedown behavior characterizes steady-state deformation, whereas for materials with low $E^*/Y$, plastic deformation continues to occur with increasing indentation cycles, indicating that plastic work is dissipated in all subsequent indentation cycles. Repeated indentation of elastic–perfectly plastic materials with low $E^*/Y$ shows the existence of elastic deformation, shakedown, and ratcheting regions, with plastic strain accumulation occurring near the center of the contact region without the growth of the plastic zone, as opposed to continuous plastic deformation in the vicinity of the contact edge and plastic zone growth with increasing indentation cycles for kinematic strain hardening behavior, which is attributed to the shift of the yield surface.

## 1. Introduction

Indentation of an elastic–plastic half-space by a rigid sphere is one of the most fundamental problems in contact mechanics and has numerous engineering applications. Deformation of spherical contacts represents a theoretical foundation for interpreting the mechanical properties of materials measured with probe-based techniques for a wide range of contact sizes and indentation depths (Fischer-Cripps, 2011). The mechanics of spherical contacts is of high importance to the fatigue life of various mechanical components subjected to cyclic contact loads, such as microswitches (Majumder et al., 2001) and hard-disk drives (Komvopoulos, 2000), and the development of constitutive models of single-asperity contacts (Yan and Komvopoulos, 1998; Jackson and Streator, 2006; Kadin et al., 2006a,b), which can be used to analyze the complex contact behavior of real rough surfaces.

Although elastic–plastic indentation of a half-space has been considered in several analytical (Galanov, 1981; Borodich, 1989, 1993; Johnson, 1985; Hill et al., 1989;

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Biwa and Storåkers, 1995), semi-analytical (Jacq et al., 2002; Nélias et al., 2006; Chen et al., 2008), and numerical (Bhattacharya and Nix, 1988; Komvopoulos and Ye, 2001; Kogut and Etsion, 2002; Song and Komvopoulos, 2013) contact mechanics studies, relatively few investigations have been devoted to the unloading response of an indented elastic–plastic half-space. One of the primitive analytical models of elastic–plastic indentation unloading is attributed to Johnson (1985), who assumed purely elastic behavior during unloading and expressed the indentation depth at the center of a spherical contact in terms of the contact force and effective elastic modulus. This model is in good qualitative agreement with Tabor's (1948) observations of plastic indentation of a soft metal by a hard steel ball. Mesarovic and Johnson (2000) also assumed elastic behavior during unloading in order to study the evolution of contact area and contact pressure distribution during unloading after fully-plastic deformation during indentation loading, and using a rigid-punch decomposition method, they derived a closed-form solution of the pressure distribution, which asymptotically approaches the Hertzian solution with decreasing contact area. Li and Gu (2009) examined the unloading behavior of two contacting bodies possessing nonlinear surface profiles and obtained analytical solutions of the contact force, normal displacement, and contact pressure distribution.

Kogut and Komvopoulos (2004) examined the unloading behavior of an elastic–perfectly plastic half-space indented by a rigid sphere for a wide range of the effective elastic modulus-to-yield strength ratio and found a correlation between the recovery of normal displacement at the center of contact and the ratio of the elastic energy released upon full unloading to the total work dissipated during loading. Etsion et al. (2005) used a finite element model to analyze the unloading response of an elastic–plastic sphere in frictionless contact with a rigid plane and obtained constitutive equations of the contact force and contact radius for unloading by curve fitting the numerical results. Zait et al. (2010) extended the former analysis to full-stick contact and observed an increase in contact area compared to frictionless contact, which was attributed to the restriction of elastic recovery in the radial direction during unloading.

Kral et al. (1993) examined elastic–plastic deformation in a half-space due to repeated indentation by a rigid sphere and obtained finite element solutions of the surface and subsurface stresses. In that study, a spherical band of tensile hoop stress was found to extend from the axis of symmetry to the surface, preventing plastic zone growth. Kral et al. (1993) also reported re-yielding in the surface region just outside the contact area upon initial unloading, further yielding in subsequent indentation cycles within the plastic zone formed during the first indentation cycle, and rapidly decreasing increment of average plastic strain with increasing indentation cycles, indicating the occurrence of shakedown at steady state. Yan and Li (2003) used the finite element method to analyze contact between a rigid sphere and an elastic–perfectly plastic half-space and showed that the contact pressure distribution increasingly deviated from the Hertzian solution with increasing loading cycles. Kadin et al. (2006a,b) analyzed multiple indentation loading and unloading of an elastic–plastic spherical contact for a wide range of material properties and observed re-yielding within a circumferential region close to the contact edge upon initial unloading and plastic flow in the same region when the maximum normal displacement exceeded a threshold value, which increased with Poisson's ratio and strain hardening exponent.

Despite important insight into the unloading behavior of indented materials provided by earlier studies, a general constitutive model of the unloading response in elastic–plastic spherical indentation was not provided and the role of strain hardening on the unloading behavior was not thoroughly examined. Moreover, knowledge of the evolution of plasticity in elastic–perfectly plastic and strain hardening (especially kinematic hardening) materials subjected to repeated indentation is fairly limited. Hence, the principal objective of this study is to provide constitutive equations of the unloading response in spherical indentation accounting for strain hardening and to examine the accumulation of plasticity due to repeated spherical indentation by tracking the development of the plastic zone and the evolution of plastic work in each indentation cycle.

## 2. Contact model

Fig. 1 schematically shows a half-space in contact with a rigid sphere of radius $R$ under load and after full unloading. Under contact load $P$, the rigid sphere penetrates the half-space to indentation depth $\delta$, conforming to the half-space over a circular contact area $A$ of radius $r$ (Fig. 1(a)). Plastic deformation during indentation loading yields a residual impression of maximum depth $\delta_{res}$ upon full unloading (Fig. 1(b)). Quasi-static indentation loading was simulated by incrementally advancing the rigid sphere into the finite

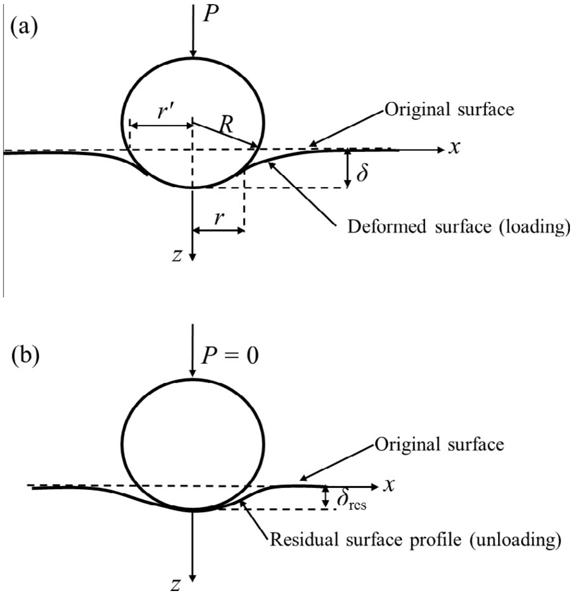

**Fig. 1.** Schematic illustration of spherical indentation of a half-space: (a) loading and (b) full unloading.