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Geometries of inhomogeneities with minimum field concentration

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This work is dedicated to Lewis Wheeler with respect and admiration on the occasion of his 73rd birthday.

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ABSTRACT

This paper is devoted to the study of geometries of inhomogeneities with minimum strain or stress concentration. The solutions are achieved by the indirect method of first deriving lower bounds and then constructing the geometries to attain the lower bounds. In particular, we show that a new class of geometries, namely, E-inclusions and periodic E-inclusions, are the optimal geometries with minimum field concentrations. We also obtain the explicit relation between the shape matrix of E-inclusion and remote applied strain which will be convenient for engineering applications of these new geometries.

1. Introduction

The failure criteria of materials are often formulated in terms of "yield stress" or "ultimate stress", meaning that the maximum stress sustained by the material cannot exceed these critical values. As is well-known, inhomogeneities such as holes or inclusions inevitably increase the local stress and strain in an elastic body (Wheeler and Kunin, 1982; Mura, 1987; Cherkaev et al., 1998; Nemat-Nasser and Hori, 1999; Vigdergauz, 2006). On the other hand, it is necessary to introduce inhomogeneities such as holes for adaptivity or desired geometry. For instance, it is common to use rivets or bolts to assemble small structural members into large, sometimes gigantic, structures such as airplanes, buildings and bridges. Also, second-phase precipitates often emerge for the coexistence of different phases of the same materials whose microstructure may be engineered to improve mechanical properties of the material (Schneider et al., 1997; Jou et al., 1997). In microelectronics, a similar dilemma occurs. To miniaturize microelectronic devices, it is desirable to use smaller conducting interconnects for realizing desired functionality. However, nuclei migrates under the bombardment of electric currents or flow of electrons and under certain critical currents or the driving force on the electrons (i.e., electric field), the migrations of nuclei become so severe that the material fails permanently (Christou, 1994).

From the above examples, it is clear that for practical engineering one needs to balance between lowering the magnitude of local fields such as stress, strain or electric field and maintaining the functionality or fulfilling the geometric constraints among others. Therefore, a precise analysis of *field concentration* is critical for the safety and reliability of the overall structure. In order to maintain the fields within safe limits, we are interested in the optimization problems of minimizing field concentration with respect to the geometries of inhomogeneities. A dimensionless quantity, namely, the *field concentration factor*, may be introduced to evaluate the severity of local field concentration in the body. Then a generic design problem

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is to find the optimal geometries of inhomogeneities such that the field concentration factor is minimized.

From a mathematical viewpoint, the dependence of field concentration on the geometries of inhomogeneities is rather complicated; one has to *a priori* solve the governing partial differential equation to evaluate the concentration factor for given geometries. In other words, the concentration factor depends *nonlocally* on the geometries of inhomogeneities. Therefore, the prevailing direct method of calculus of variation is not applicable. A conventional approach to such optimal design problems is based on an iterative process: trial geometries of inhomogeneities are chosen, the field concentration is evaluated upon a full solution of the underlying boundary value problem, and then a change of geometries is proposed to lower the field concentration via a sensitivity analysis (Haftka and Grandhi, 1986; Allaire and Jouve, 2008). This process is iterated until a local minimum of field concentration is achieved. This approach is computationally intensive and the final result, though could be satisfactory for a target application, cannot give a definitive answer to the global minimum. For the global minimum, one has to use the indirect method of first finding a lower bound on the concentration factor and then construct geometries to attain the lower bound.

In the context of linear elasticity, the problem of minimum stress or strain concentration has been discussed and reviewed by Sternberg (1958) and Wheeler (1992). Recently, there has been significant progress on a general theory concerning minimum field concentration for general measures of local fields that include the local Von Mises stress and strain (Alali and Lipton, 2009), hydrostatic stress and strain (Lipton, 2005, 2006), and local mixed modes of stress and strain (Alali and Lipton, 2012). The theory has also been established in much broader physical contexts including thermo-elastic composites (Chen and Lipton, 2012) and conductive composites (Lipton, 2003, 2004). The existing results concerning optimal geometries clearly suggest that the uniformity of field in the inhomogeneities is intimately related with the optimality of the geometries for minimum field concentration. Also, the optimal microstructures such as coated ellipsoids that achieve minimum field concentration, under suitable algebraic assumptions about the material properties, turn out to be optimal microstructures attaining the Hashin–Shtrikman's bounds of the effective properties of two-phase composites. As shown in recent works of Liu et al. (2007) (accepted), a new class of geometries, namely, E-inclusions and periodic E-inclusions,[1] have similar uniformity property as ellipsoids and achieve the Hashin–Shtrikman bounds for composites. One may wonder if they are also the optimal geometries that minimize the field concentrations. Our main goal here is to report that the answer to the above question is affirmative. We also find explicitly the relationship between the average applied strain $\bar{\mathbf{E}}$ and the shape matrix $\mathbf{Q}$ of the E-inclusions with minimum strain or stress concentration (cf., (40)). Since the shape matrices of E-inclusions have to be positive semi-definite, E-inclusions being the solutions requires that the average applied strain has to satisfy some algebraic conditions. Beyond this region, the reader is referred to Cherkaev et al. (1998) and Vigdergauz (2006, 2008) for approximate solutions and important insight.

The paper is organized as follows. In Section 2 we formulate and state the mathematical optimization problem in the context of linear elasticity. The formulation allows for simultaneous consideration of finite many inhomogeneities and periodic array of inhomogeneities and in both two and three dimensions. The lower bounds for stress and strain concentration factors are derived in Section 3. In Section 4 we show that E-inclusions indeed achieve the lower bounds of minimum stress or strain concentration. We conclude and provide an outlook of potential engineering applications in Section 5.

Notation. Since stress and strain are symmetric tensor fields, we introduce the following l^p-norm of a symmetric matrix $\mathbf{M} \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ for $p \in [1,\infty]$:

$$\|\mathbf{M}\|_{l^p} := \left(\sum_{i=1}^{n} |\lambda_i(\mathbf{M})|^p\right)^{1/p}, \tag{1}$$

where $\lambda_1(\mathbf{M}) \leqslant \cdots \leqslant \lambda_n(\mathbf{M})$ are the ordered eigenvalues of the symmetric $\mathbf{M}$. We remark that $\|\mathbf{M}\|_{l^p} = (\mathbf{M}\cdot\mathbf{M})^{1/2}$ is the usual Euclidean norm for $p = 2$, $\|\mathbf{M}\|_{l^p} = \sum_{i=1}^{n} |\lambda_i(\mathbf{M})|$ if $p = 1$, and $\|\mathbf{M}\|_{l^p} = \max\{|\lambda_i(\mathbf{M})| : i = 1,\ldots,N\}$ if $p = \infty$.

2. Problem statement

Consider an infinite homogeneous elastic body occupying the entire Euclidean space $\mathbb{R}^n (n = 2$ or $3)$. Let $\mathbf{C}_0 : \mathbb{R}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}^{n\times n}_{\mathrm{sym}}$ be the fourth-order stiffness tensor of the body, $\mathbf{u} : \mathbb{R}^n \to \mathbb{R}^n$ be the displacement, and $\boldsymbol{\sigma} : \mathbb{R}^n \to \mathbb{R}^{n\times n}_{\mathrm{sym}}$ be the stress. Assume that the body is under the application of an average strain for some $\bar{\mathbf{E}} \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$:

$$\mathbf{u}(\mathbf{x}) = \bar{\mathbf{E}}\mathbf{x} + O(1) \quad \text{as } |\mathbf{x}| \to +\infty. \tag{2}$$

In the absence of body force, the equilibrium state of the body requires that

$$\mathrm{div}\,\boldsymbol{\sigma} = 0 \quad \text{in } \mathbb{R}^n. \tag{3}$$

Also, it is clear that a solution to the above equation with the boundary condition (2) is given by

$$\mathbf{u} = \mathbf{u}_0 := \bar{\mathbf{E}}\mathbf{x} \quad \text{in } \mathbb{R}^n.$$

Let $Y \subset \mathbb{R}^n$ be a "representative volume element" of the body. We now introduce N mutually disjoint inhomogeneities $\Omega_\alpha \subset Y\ (\alpha = 1,\ldots,N)$ of materials with stiffness tensor $\mathbf{C}_\alpha$. Two scenarios will be considered: (i) the inhomogeneities are distributed in a bounded region in $\mathbb{R}^n$, and (ii) the inhomogeneities are distributed periodically in the whole space $\mathbb{R}^n$. The representative volume element Y is taken as the entire space $\mathbb{R}^n$ for the former case whereas, without loss of generality, can be assumed to be $Y = (0,1)^n$ for the latter case. We remark that the latter case corresponds to a periodic composite with infinitely many inhomogeneities occupying $\{\Omega_\alpha + \sum_{i=1}^{n} k_i\mathbf{f}_i : \alpha = 1,\ldots,N;\ k_1,\ldots,k_n$ are integers$\}$. $(\mathbf{f}_1,\ldots,\mathbf{f}_n$ is the basis of our rectangular

[1] E-inclusions or periodic E-inclusions in two dimensions are first constructed by Vigdergauz (1976, 1986).