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# Fibrous composites of piezoelectric and piezomagnetic phases: Generalized plane strain with transverse electromagnetic fields

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This work presents a theoretical framework for solving the field distributions of a piezoelectric–piezomagnetic fibrous composite subjected to generalized plane strain with transverse electromagnetic fields. The matrix is infinite containing arbitrarily distributed circular cylinders, which may have different sizes and material properties. By introducing an eigenstrain corresponding to the electro-magneto-elastic effect, this coupling problem can be reduced to an equivalent plane elasticity problem. The classic work of Muskhelishvili to obtain the elastic potential of a composite is generalized to the current multi-field multi-inclusion media. Several numerical examples are presented to demonstrate the effectiveness of the approach.



## 1. Introduction

The magneto-electric (ME) coupling refers to the polarization induced by a magnetic field, or conversely the magnetization induced by an electric field. It was first predicted by Landau and Lifshitz (1984) and observed by Astrov (1960) and Rado and Folen (1961) over fifty years ago. This ME effect has recently drawn ever-increasing interest due to their potential applications as multifunctional devices including ME data storage and switching (Spaldin and Fiebig, 2005), modulation of optical waves (Fiebig, 2005), and electrically microwave phase shifters (Bichurin et al., 2002). However, the coupling is rather weak in a single-phase material even at low temperature, and this has motivated the study of composites of piezoelectric and piezomagnetic media. The "product property" causes the ME effect in this multiferroic composite: an applied electric field generates a deformation in the piezoelectric phase, which in turn generates a deformation in the piezomagnetic phase, resulting a magnetization (Nan et al., 2008).

The promise of applications, and the indirect coupling through strain have also made ME composites the topic of a number of theoretical investigations. Among them, Nan (1994), Srinivas and Li (2005) and Liu and Kuo (2012) estimated the effective properties of ME composites of non-dilute volume fractions by mean-field-type models. Benveniste (1995) derived exact relations in a ME composite with cylindrical geometry. The analysis for local fields is available for simple microstructures such as a single ellipsoidal inclusion (Huang and Kuo, 1997; Li and Dunn, 1998), arbitrarily distributed or periodic arrays of fibrous ME composites (Kuo, 2011; Kuo and Pan, 2011; Kuo and Bhattacharya, 2013), and laminates (Kuo et al., 2010). In addition, Liu et al. (2004) and Lee et al. (2005) used finite element method to address ME composites for general microstructures, while Aboudi (2001) and Camacho-Montes et al. (2009) adopted the homogenization method. However, much of this work uses approximate methods and models based on single inclusions to estimate the

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effective properties of composites. Exact methods that provides the detailed field distribution are limited to the medium subjected to the anti-plane shear with in-plane electromagnetic fields due to the complexity.

In a classic work, Muskhelishvili (1975) used the Kolosov–Muskhelishvili potentials with truncated Laurent series for elastic problems with circular boundaries. Analogous representations were employed by McPhedran and Movchan (1994) for a pair and a square array of circular elastic inclusions, by Buryachenko and Kushch (2006) for a matrix reinforced two linearly elastic isotropic aligned circular fibers, and by Kushch et al. (2008) for the progressive damage in the fiber reinforced composite. This method was extended to investigate the multiple piezoelectric inclusions in a non-piezoelectric matrix (Yang and Gao, 2010), and for a three-phase thermo-electro-magneto-elastic cylinder model (Tong et al., 2008). In addition, a Galerkin boundary integral method has also been developed to address the elastic composites with multiple circular cylinders (Mogilevskaya and Crouch, 2001), while Eshelby's equivalent inclusion for a fibrous piezoelectric inhomogeneity was proposed by Xiao and Bai (1999). In this paper, we generalize Muskhelishvili's methodology to a fibrous composite made of piezoelectric and piezomagnetic phases under generalized plane strain ($\varepsilon^0_{13}=0, \varepsilon^0_{23}=0, \varepsilon^0_{33}\neq 0$) with transverse electromagnetic fields. Specifically we seek the stress and displacement distributions of the composite.

The remainder of this paper is organized as follows. In Section 2 we formulate the equation for a piezoelectric–piezomagnetic composite under generalized plane strain with transverse electromagnetic fields. We show that the multi-field coupled problem can be reduced to an equivalent plane elastic problem with a corresponding uniform eigenstrain. In Section 3 we generalize the work of Muskhelishvili (1975) to obtain a representation of the solution. The basic idea is to express the stress and displacement via two complex potentials and expand each field in each medium in a series. We use this method to study selected systems with sufficient accuracy in Section 4.

## 2. General framework

Let us consider an infinite medium containing $N$ arbitrarily distributed, parallel and separated circular cylinders (Fig. 1). The domain of the $p$th circular cylinder is denoted $V_p,\ p=1,2,\ldots,N$, and the remaining matrix is denoted $\Omega_m$. We assume that the cylinders and the matrix are made of distinct phases: transversely isotropic piezoelectric or piezomagnetic materials. A global Cartesian coordinate system is introduced with $x_1$- and $x_2$-axes in the plane of the cross-section and $x_3$-along the axes of the cylinders (Fig. 1). The centers of the $p$th circular cylinder are designated as $O_p$, each of which may have different radii $a_p$.

Assume that the composite is subjected to in-plane mechanical strain $\varepsilon^0_{11},\ \varepsilon^0_{22}$ and $\varepsilon^0_{12}$ (or in-plane stress $\sigma^0_{11},\ \sigma^0_{22}$ and $\sigma^0_{12}$) at infinity and uniform strain $\varepsilon^0_{33}$, electric field $E^0_3$ and magnetic field $H^0_3$ in the $x_3$-direction. It can be shown that the general constitutive law for the non-vanishing

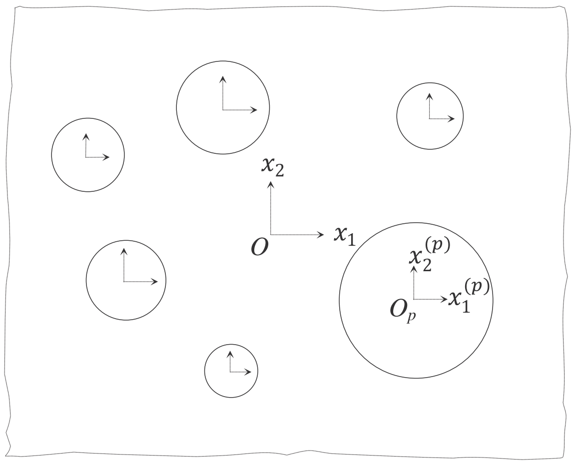


**Fig. 1.** The cross-section of the multiple fibers composite.

field quantities can be written in a compact form as (Benveniste, 1995)

$$\begin{pmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{12}\\ D_3\\ B_3\end{pmatrix}=\begin{pmatrix}C_{11} & C_{12} & C_{13} & 0 & e_{31} & q_{31}\\ C_{12} & C_{11} & C_{13} & 0 & e_{31} & q_{31}\\ C_{13} & C_{13} & C_{33} & 0 & e_{33} & q_{33}\\ 0 & 0 & 0 & C_{66} & 0 & 0\\ e_{31} & e_{31} & e_{33} & 0 & -\kappa_{33} & -\lambda_{33}\\ q_{31} & q_{31} & q_{33} & 0 & -\lambda_{33} & -\mu_{33}\end{pmatrix}\begin{pmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{12}\\ -E_3\\ -H_3\end{pmatrix}. \tag{2.1}$$

Here $\sigma_{ij}$ and $\varepsilon_{ij}$ are the stress and strain; $D_i$ and $E_i$ are the electric displacement and electric field; $B_i$ and $H_i$ are the magnetic flux and magnetic field, respectively. $C_{11},\ C_{12},\ C_{13},\ C_{33}$, and $C_{66}$ are the elastic moduli, $e_{31}$ and $e_{33}$ are piezoelectric constants, $q_{31}$ and $q_{33}$ are piezomagnetic constants, and $\kappa_{33},\ \mu_{33}$, and $\lambda_{33}$ are the dielectric permittivity, magnetic permeability and magnetoelectric coefficients, respectively.

The constitutive equation (2.1) are rather complicated. However, it is observed that $\varepsilon_{33},\ E_3$, and $H_3$ are constants in the composite (Tong et al., 2008). Thus we can introduce an uniform eigenstrain field

$$\begin{aligned}\varepsilon^* &= \varepsilon^*_{11}=\varepsilon^*_{22}\\ &=(-C_{13}\varepsilon_{33}+e_{31}E_3+q_{31}H_3)/(C_{11}+C_{12}).\end{aligned} \tag{2.2}$$

Substitution of Eq. (2.2) into Eq. (2.1) yields

$$\begin{aligned}&\sigma_{11}+\sigma_{22}=2K[(\varepsilon_{11}+\varepsilon_{22})-2\varepsilon^*],\\ &\sigma_{22}-\sigma_{11}=2\mu(\varepsilon_{22}-\varepsilon_{11}),\\ &\sigma_{12}=2\mu\varepsilon_{12}\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}&\sigma_{33}=C_{13}(\varepsilon_{11}+\varepsilon_{22})+C_{33}\varepsilon_{33}-e_{33}E_3-q_{33}H_3,\\ &D_3=e_{31}(\varepsilon_{11}+\varepsilon_{22})+e_{33}\varepsilon_{33}+\kappa_{33}E_3+\lambda_{33}H_3,\\ &B_3=q_{31}(\varepsilon_{11}+\varepsilon_{22})+q_{33}\varepsilon_{33}+\lambda_{33}E_3+\mu_{33}H_3,\end{aligned} \tag{2.4}$$

where $K=(C_{11}+C_{12})/2$ is the in-plane bulk modulus, and $\mu=(C_{11}-C_{12})/2$ is the transverse shear modulus. It is