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On a rigid inclusion pressed between two elastic half spaces

Andrei Kotousov *, Luiz Bortolan Neto, Aditya Khanna

School of Mechanical Engineering, The University of Adelaide, SA 5005, Australia

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ABSTRACT

A solution to the problem of a rigid cylindrical inclusion pressed between two elastic half spaces is obtained using the distributed dislocation technique. The solution is compared with previously published analytical and numerical results for a rigid cylindrical inclusion bounded by two parabolic arcs with rounded corners. A simplified solution to the problem based on the classical contact theory and well-known results for crack problems is also suggested and validated. The simplified solution agrees well with analytical results in the case when the length of the opening around inclusion is much larger than the length of the contact zone.

1. Introduction

The problem of a rigid inclusion pressed between two elastic half spaces has many important applications, for example, in the investigation of friction properties of flat surfaces in the presence of contamination, in the design of various engineering components like fasteners and in various particle technologies such as hydraulic fracturing which involve the injection of small particles (proppant) into artificial fractures or natural cracks present in oil/gas bearing rocks.

The two dimensional problem of a crack opened by a rigid inclusion was first considered by Lowengrub and Srivastav (1970). These researchers used the theory of dual and triple integral equations based on Fourier transform techniques of Sneddon (1957) and obtained a solution by assuming that the length of the contact between the inclusion and the solid body is known. However, in these types of problems, the contact area is generally not known in advance and has to be determined from the smooth tangency of the crack surface at the inclusion ends (Barenblatt, 1962; Cherepanov, 1979). Maiti (1980) reduced the problem with unknown equilibrium contact length to a Föppl integral equation, for which the solution can be easily found in closed form, see Tricomi (1985). However, in the problems considered by Lowengrub and Srivastav (1970) and Maiti (1980), the elastic spaces were assumed to be free from stresses at infinity, so that the opened area and resultant stresses are due to the presence of the rigid inclusion only.

Alblas presented a closed-form solutions to the problem of a rigid cylindrical inclusion, bounded by two parabolic arcs with rounded corners and pressed between two identical elastic half spaces (Alblas, 1974) or two elastics layers of equal thickness (Alblas, 1975). The unknown contact region is calculated by solving a transcendental equation involving elliptic integrals. An approximate solution to these problems based on the Chebyshev polynomials was presented by Gladwell (1977). The latter solution converges to Alblas' closed form analytical solution when the depth of the elastic layers is roughly twice larger than the length of the opening. The numerical approach developed by Gladwell was specifically suitable for the parabolic shape of the rigid inclusion as many of the integrals involved in the solution may be computed explicitly. However, both solutions experience computational difficulties at a relatively small length of the contact between the

* Corresponding author.
E-mail address: andrei.kotousov@adelaide.edu.au (A. Kotousov).

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inclusion and half spaces, which corresponds to large opening areas or relatively thin elastic layers (Gladwell, 1977).

A related axisymmetric problem in which a rigid oblate spheroidal inclusion is pressed between two dissimilar elastic half-spaces was considered by Gladwell and Hara (1981). The problem is quite relevant to the analysis of a rigid spherical proppant pressed between the faces of a crack in an oil/gas bearing rock; however the analysis of Gladwell and Hara (1981) was based on the assumption that the contact radius between the inclusion and the elastic half space is known in advance. An axisymmetric inclusion problem with an unknown equilibrium contact length between the half spaces was first solved by Selvadurai (1993) in terms of Hankel transforms for a rigid disc-shaped inclusion of constant thickness. The analysis was intended to serve as a simplified model for the fracture and proppant interaction scenario (Selvadurai, 1993, 1994). Selvadurai (1993) also presented an alternative method in which the problem of a rigid disc inclusion between two elastic half-spaces was decomposed into two auxiliary problems which are: (1) the problem of a penny shaped crack opened by rigid disc inclusion and (2) an annular crack subject to uniform tensile stress. The two methods were found to be in excellent agreement.

Several other similar problems have been considered more recently. These include the problem of a non-axisymmetric inclusion of constant thickness pressed between elastic half-spaces (Gladwell, 1995), the indentation of a pre-compressed penny shaped crack by a rigid disc (Selvadurai, 2000), the separation of dissimilar elastic half spaces due to axisymmetric stress fields (Selvadurai, 2003) and the separation of dissimilar piezoelectric half spaces by a rigid disc inclusion (Eskandari et al., 2009).

In this paper, the class of plane strain problems initiated by Alblas (1974) is revisited and two solution techniques are described for the problem of a rigid cylindrical inclusion pressed between two identical half-spaces. Firstly, an approximate solution to the problem is obtained based on the assumption that the length of the opened region between the half spaces is much larger than the size of the inclusion. In this simplified solution, it is also assumed that the stress distribution over the zone of contact is described by the classical Hertz theory of contact stresses between a rigid circular cylinder and an elastic half space (Johnson, 1985). Secondly, a method is developed based on the approach of Maiti (1980), which is reformulated in terms of unknown distributed dislocation densities (Codrington and Kotousov, 2007; Hills et al., 1996). The solution is obtained by the superposition of analytical results for the Föppl integral equation and the analytical solution for two collinear cracks in an infinite plate subjected to uniform remote stress on infinity given by Willmore (1949). This approach is analogous to Selvadurai's solution for disc shaped inclusions. The two solution methods are compared with the previously published analytical and numerical results for rigid inclusions of parabolic shape. The dislocation solution also confirms the applicability of the simplified approximate solution in the case of large openings around parabolic and circular rigid inclusions.

2. Problem formulation

Consider a rigid cylindrical inclusion with the shape, which permits a snug and smooth contact, squeezed by two semi-infinite elastic spaces. The elastic spaces are subjected to remote compressive stress, σ_o as illustrated in Fig. 1. As a result of elastic deformations, an area $(|x| \leqslant a, y = 0)$ between two elastic half spaces is open. The shape of the inclusion determines the profile of the opening over the contact region between the spaces and the inclusion given by $|x| \leqslant c$. Due to absence of the cohesion between the elastic half spaces and between the half spaces and inclusion, there must be no stress singularities in the problem solution, in particular at the end of the inclusion contact zone $(|x| = c)$ and at the point where the half spaces join together $(|x| = a)$. Moreover, the half opened length a, and the length of contact c are not known in advance and have to be determined from the solution of the problem.

The solution to the problem can be obtained as a superposition of the applied stress $\sigma_{yy} = \sigma_o$ and a corrective solution, which negates the induced stresses along the opening, $c < x < a$ (Codrington and Kotousov, 2007; Hills et al., 1996). The corrective solution in the case of frictionless contact can be found from the consideration of the following mixed value boundary-value problem:

$$\delta_y(x,0) = 2u_y(x,0) = f(x), \quad |x| \leqslant c, \tag{1a}$$

$$u_y(x,0) = 0, \quad |x| > a, \tag{1b}$$

$$\sigma_{yy}(x,0) = -\sigma_o, \quad c < |x| < a, \tag{1c}$$

$$\sigma_{xy}(x,0) = 0 \quad |x| < \infty, \tag{1d}$$

where $\delta_y(x, 0) = \delta_y(x)$ is the crack opening.

The formulated boundary-value problem (1) can be significantly simplified if one assumes that the length of the contact area is much smaller than the characteristic size of the inclusion and the length of the opening. In this case, in accordance with the Saint–Venant principle, the distribution of the contact stresses between elastic half spaces and the rigid inclusion has to follow the classical Hertz theory (Johnson, 1985). An approximate solution based on these simplifications will be developed next.

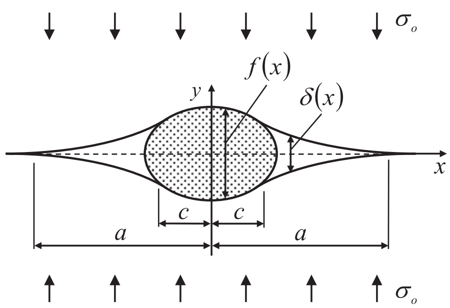

Fig. 1. Problem geometry and coordinate system.