Mechanics of Materials 68 (2014) 104–119

Contents lists available at ScienceDirect

Mechanics of Materials

journal homepage: www.elsevier.com/locate/mechmat

Analysis of intrinsic stability criteria for isotropic third-order Green elastic and compressible neo-Hookean solids

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ARTICLE INFO

Article history:
Received 5 June 2012
Received in revised form 8 August 2013
Available online 23 August 2013

Keywords:
Nonlinear elasticity
Internal stability
Third-order elastic constants
Isotropy
Polycrystals

ABSTRACT

Internal stability of isotropic nonlinear elastic materials under homogeneous deformation is studied. Results provide new insight into various intrinsic stability measures, first proposed elsewhere, for generic nonlinear elastic solids. Three intrinsic stability criteria involving three different tangent elastic stiffness matrices are considered, corresponding to respective increments in strain measures conjugate to thermodynamic tension, first Piola–Kirchhoff stress, and Cauchy stress. Primary deformation paths of interest include spherical (i.e., isotropic) deformation, uniaxial strain, and simple shear; unstable modes are not constrained to remain along primary deformation paths. Effects of choices of second- and third-order elastic constants on intrinsic stability are systematically studied for physically realistic ranges of constants. For most cases investigated here, internal stability according to strain increments conjugate to Cauchy stress is found to be the most stringent criterion. When third-order constants vanish, internal stability under large compression tends to decrease as Poisson's ratio increases. When third-order constants are nonzero, a negative (positive) pressure derivative of the shear modulus often promotes unstable modes in compression (tension). For large shear deformation, larger magnitudes of third-order constants tend to result in more unstable behavior, regardless of the sign of the pressure derivative of the shear modulus. A compressible neo-Hookean model is generally much more intrinsically stable than second- and third-order elastic models when Poisson's ratio is non-negative.

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1. Introduction

Stability of elastic solids under finite deformation has been the subject of numerous studies, with early work on intrinsic stability of crystals due to Born (1940). Nonlinear elastic anisotropic solids (e.g., single crystals) have been analyzed in a number of works (Hill, 1975; Hill and Milstein, 1977; Milstein and Hill, 1979; Wang et al., 1993, 1995; Morris and Krenn, 2000), as have nonlinear elastic isotropic solids (Hill, 1957; Rivlin, 1974; Rivlin and Beatty, 2003).

Various criteria for stability of elastic solids have been proposed in the literature, beginning with work of Born (1940) who associated stability with a positive definite stiffness measure and local convexity of internal energy expressed in terms of a Lagrangian Green strain measure. In a real physical system, the appropriate choice of stability criterion depends on the method of static incremental load application (e.g., dead loading in one or more directions (Rivlin, 1974)) and any constraints associated with boundary conditions, and such a criterion may not correspond to Born's. When considering "intrinsic" or "internal" stability of a unit cell or unit cube of a given material, from the continuum viewpoint or using atomic theory, the proper choice of stability measure is ambiguous when the precise loading mechanism is left unspecified. Different choices of conjugate stress–strain measures (i.e., different generalized coordinates and conjugate forces) can lead to different

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0167-6636/$ - see front matter Published by Elsevier Ltd.
http://dx.doi.org/10.1016/j.mechmat.2013.08.007

local convexity conditions and different intrinsic stability criteria (Hill, 1975; Hill and Milstein, 1977). As stated by Milstein and Hill (1979), consistency of classical, unique, and environment-dependent stability criteria with intrinsic stability or convexity arguments such as Born's requires a special environment in which loads can be varied to follow the material during any disturbances while fixing values of conjugate forces, which in turn are non-unique since they depend on the choice of generalized coordinates (i.e., the choice of strain measure). This load environment used to probe stability (e.g., via virtual deformations at a fixed stress) need not correspond to the loading program used to achieve the stressed equilibrium state from which stability is tested (Hill and Milstein, 1977; Milstein and Hill, 1979).

Intrinsic stability requirements considered in the present work lead to restrictions on various incremental tangent elastic moduli; stability measures of this sort applied to superposed deformation variations in any direction may require that an energy function be locally convex, or at least that a particular stiffness matrix (i.e., Hessian) be positive definite, at the current equilibrium state. For nonlinear materials of high symmetry (e.g., isotropic solids) under simple deformation paths, such intrinsic stability requirements can often be stated succinctly in terms of restrictions on a few strain-dependent elastic coefficients. In linear elastic solids, such requirements degenerate to the usual constitutive constraints of positive bulk and shear moduli. As noted above and explained in Hill and Milstein (1977) and Parry (1978), intrinsic stability measures like those considered herein do not depend on the load environment, but do depend on the choice of conjugate stress–strain measures. Comparisons among intrinsic criteria involving different Lagrangian strain measures were derived for generic isotropic elastic solids under positive principal stresses (Parry, 1978).

The distinction between material instability and structural instability should be noted. As defined in the present work, material instability correlates with intrinsic instability, and depends only on material properties and loading protocol. Intrinsic stability criteria are local since they consider only homogeneous stress/deformation states up to the onset of instability. In nonlinear materials (e.g., nonlinear elastic, elastic–plastic, or damaged solids), the onset of material instability depends on strain, but in linear elastic solids, material stability is independent of strain and simply requires positive definiteness of the tensor of elastic constants. In contrast, structural instability criteria are global rather than local, depending on geometry of the body. The stress/strain state may be inhomogeneous prior to onset of structural instability. Structural instability may occur even if the material is intrinsically stable, and can be induced by loads even in linear elastic materials. A representative example is buckling of a slender column under compression.

Intrinsic stability properties of solids under large compressive stress or pressure are of interest for applications in ballistics, impact phenomena, and earth and planetary sciences. Elastic instability may signal the onset of failure or localization phenomena, e.g., slip, fracture, or phase transformations (Hill, 1975; Wang et al., 1993; Morris and Krenn, 2000). Hard materials of lower symmetry such as quartz (Gregoryanz et al., 2000), silicon carbide (Clayton, 2010), and boron carbide (Clayton, 2012) exhibit a decrease in certain shear elastic stiffness components with increasing pressure. At high pressures, this tendency may lead to the onset of instability and subsequent amorphization (Chen et al., 2003), which in the case of ceramic materials hinders performance in ballistic applications. In contrast to these polyatomic ceramic materials, most elemental engineering materials demonstrate increasing shear moduli with increasing pressure (Guinan and Steinberg, 1974), which would tend to enhance rather than diminish internal stability at large compressions.

Nonlinear elastic models of anisotropic single crystals (Wallace, 1972; Teodosiu, 1982; Clayton, 2011) typically assume a strain energy function written as a Taylor polynomial in Green (Lagrangian) elastic strain $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F} - \mathbf{1})$, with $\boldsymbol{F}$ the deformation gradient. Such a model, when terms of up to third order are maintained (i.e., second- and third-order elastic constants) provides reasonably accurate descriptions of stresses and wave propagation for moderate compressions (Thurston, 1974; Clayton, 2009); however, Eulerian strain measures may be more accurate for extreme pressures (Weaver, 1976; Jeanloz, 1989).

The present work focuses primarily on isotropic elastic solids of third order. Such materials are described by two independent second-order elastic constants and three independent third-order elastic constants (Murnaghan, 1937; Teodosiu, 1982). Because of the limited number of constants, systematic study of effects of choices of constants on intrinsic stability for simple monotonic deformation paths is tractable, and is undertaken in this work. Third-order constants are related explicitly to pressure derivatives of bulk and shear moduli in the reference state (Thurston et al., 1966; Guinan and Steinberg, 1974; Teodosiu, 1982); experimental data (Guinan and Steinberg, 1974; Steinberg, 1982) thus provide realistic bounds on combinations of third-order constants. Choices of third-order constants yielding a decreasing shear stiffness with increasing compression provide insight into behavior of aforementioned ceramic materials demonstrating shear instabilities at high pressure (Gregoryanz et al., 2000; Chen et al., 2003). It is noted that single crystals of such materials are highly anisotropic (e.g., trigonal symmetry: six second-order and fourteen third-order elastic constants) and systematic study of effects of varying all elastic constants individually on stability is intractable. For comparison, intrinsic stability of a class of compressible neo-Hookean solids (Simo and Pister, 1984), which demonstrate a strongly increasing bulk modulus with pressure, is also considered.

This paper is organized as follows. Requisite quantities associated with internal stability are derived in Section 2. Intrinsic stability of third-order elastic solids in terms of three different criteria from the literature (Born, 1940; Hill, 1975; Wang et al., 1993) is analyzed in Section 3. For each criterion, minimum eigenvalues of a particular tangent stiffness matrix are examined for different choices of second- and/or third-order elastic constants for an element of material undergoing spherical deformation, uniaxial