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Plastic collapse of lattice structures under a general stress state

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ABSTRACT

We present a numerical minimization procedure to determine the macroscopic 'plastic collapse strength' of a tessellated cellular structure under a general stress state. The method is illustrated with sample cellular structures of regular and hierarchical honeycombs. Based on the deformation modes found by minimization of plastic dissipation, closed-form expressions for strength are derived. The current work generalizes previous studies on plastic collapse analysis of lattice structures, which are limited to very simple loading conditions.

1. Introduction

Strength and energy absorption capacity of lattice structures are governed by buckling of the cell walls or plastic yielding of cell wall material (Evans et al., 2001; Evans et al., 1998; Ashby, 2006; Papka and Kyriakides, 1994; Jang and Kyriakides, 2009; Babaee et al., 2012). However, the current state of the literature on collapse of cellular lattice structures is limited to structures subjected to simple loading conditions such as uniaxial, biaxial or shear loading applied at special orientations (Gibson et al., 1989; Gibson et al., 1982; Haghpanah et al., 2013; Onck et al., 2001; Karagiozova and Yu, 2004; Zhu and Mills, 2000). In this article, we focus on plastic deformation, and present a method that allows numerical and algebraic calculation of plastic collapse strength under arbitrary states of stress or strain. The presented method is based on minimizing the internal plastic dissipation inside a unit cell of the tessellated structure (Chen et al., 2007). To illustrate the method, two two-dimensional networks of rigid-plastic beams are considered. First, a hexagonal network (*honeycomb*) with sixfold rotational symmetry, and second, the first iteration of the honeycomb structure in a hierarchical refinement scheme in which all three–edge nodes are replaced with smaller, parallel hexagons defined by size ratio γ, with $\gamma = 0$ denoting a regular honeycomb. The latter structure, also maintaining a microscopic sixfold symmetry, is called *first order hierarchical honeycomb* (Ajdari et al., 2012), see Fig. 1. The relations between macroscopic stress and strain states and unit cell reaction forces and displacements, respectively, are derived in Section 2 in a convenient canonical position which is suitable for threefold symmetrical structures. The minimization of plastic dissipation inside the unit cell subjected to external forces or displacements is detailed in Section 3. We then exploit the observed unit-cell deformation patterns to derive analytical expressions for strength, to permit efficient computation and plotting. The results from the minimization scheme and also the derived upper bounds of plastic collapse are presented in Section 4. Lastly in Section 5, a summary of the current work and conclusions are given.

2. Threefold definitions of stress and strain

We begin our analysis with threefold symmetric definitions of both microscopic and macroscopic stress and strain (note that for structures without a threefold symmetry (e.g. square honeycomb) the conventional Cartesian coordinate system can be used readily). To carry out the

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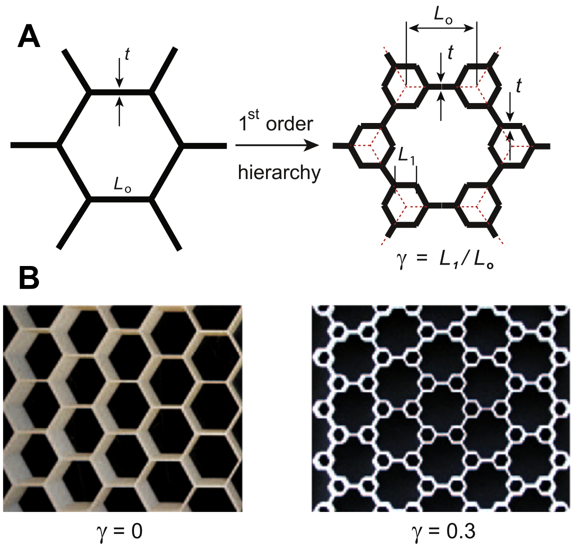

Fig. 1. (A) Schematics of regular and hierarchical honeycombs with one level of hierarchy. (B) Images of *regular* and *hierarchical* ($\gamma = 0.3$) honeycombs with $L_0 = 2$ cm fabricated using three-dimensional printing.

analyses we select a *unit cell*, with associated tractions and displacements, which tiles the plane to represent the loaded lattice structure. The structural unit cell for our hexagonal-based patterns encompasses one vertex of the original hexagonal network, out to the midpoints of the original hexagon sides (a distance $L_0/2$), see Fig. 2(A). The area associated with the unit cell is a triangle joining the three hexagon center-points that surround this vertex, with area $3\sqrt{3}L_0^2/4$. The general state of stress is expressed in terms of its normal components in the three in-plane material directions $a = 0^\circ$, $b = 120^\circ$ and $c = 240^\circ$: σ_{aa}, σ_{bb}, σ_{cc}. Given those three normal components, the *xy* stress tensor can be written:

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{xy} & \sigma_{yy} \end{bmatrix} = \begin{bmatrix} \sigma_{aa} & \frac{\sigma_{cc}-\sigma_{bb}}{\sqrt{3}} \\ \frac{\sigma_{cc}-\sigma_{bb}}{\sqrt{3}} & \frac{2\sigma_{cc}+2\sigma_{bb}-\sigma_{aa}}{3} \end{bmatrix} \quad (1)$$

where *x* and *y* axes are taken along the so-called armchair (or ribbon) and zigzag (or transverse) directions.

We then examine the allowable external loads at numbered points 1, 4, 7 of the unit cell shown in Fig. 2(B). First we argue that there are no moments applied at these points: the 180° rotational symmetry of the tessellated structure (and trivially the components of microscopic stress) means that any upwards curvature at such a point must become downwards curvature after the rotation, and the only way these can be equal is to have the value zero. Then, using the vertical cut line Δ_a which intersects horizontal sides with a spacing $L_0\sqrt{3}$, we deduce the value of radial force F_a to be $\sigma_{aa}L_0\sqrt{3}$, and similarly for radial directions b, c. Note also that the arbitrary radial forces F_a, F_b, F_c will not be in equilibrium so there must be transverse forces G_a, G_b, G_c, defined as positive counterclockwise about the origin. Successively taking moments of forces about pairwise intersections of G_a, G_b, G_c , we find $G_a = (F_c - F_b)/\sqrt{3}$, and cyclically.

Next, we consider relations between macroscopic strain and relative displacements of the unit cell boundary points. Given arbitrary radial and tangential displacements of points 1, 4, 7, we can use rigid body displacements and rotation to place the deformed unit cell uniquely in a *canonical position* with the boundary points 1, 4, 7 still on the a, b, c lines. In that unique placement, the boundary point *canonical radial displacements* along the a, b, c lines are named δ_a, δ_b, δ_c, where the segments 1–2, 4–5, 8–7 are generally no longer parallel to those lines. Since the boundary loads are in equilibrium, the introduced rigid body displacements and rotations do not affect the net work.

Given δ_a, the strain is uniaxial along a. Its magnitude is the change in the unit cell x dimension divided by the original unit cell x dimension $3L_0/4$, in other words $\epsilon_a = 4\delta_a/3L_0$. Purely uniaxial strains in all three directions can be superposed to define a general xy strain tensor:

$$\begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} \\ \epsilon_{xy} & \epsilon_{yy} \end{bmatrix} = \begin{bmatrix} \frac{4\delta_a+\delta_b+\delta_c}{3L_0} & \frac{\sqrt{3}(\delta_c-\delta_b)}{3L_0} \\ \frac{\sqrt{3}(\delta_c-\delta_b)}{3L_0} & \frac{\delta_b+\delta_c}{L_0} \end{bmatrix} \quad (2)$$

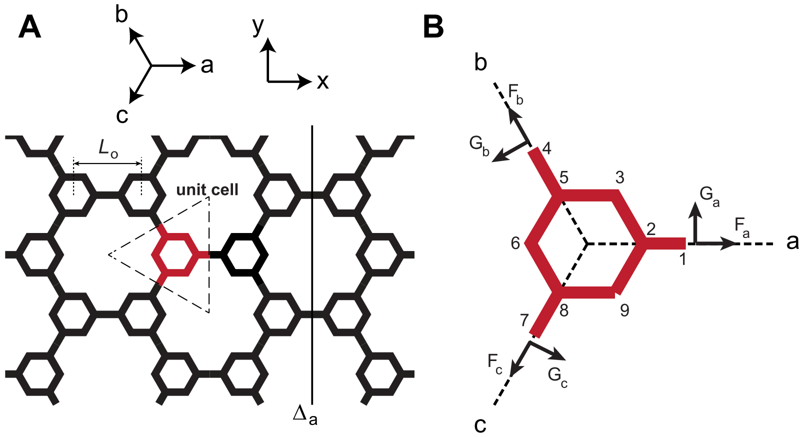

Fig. 2. (A) Schematic of a hierarchical honeycomb where a unit cell of the structure is marked by red lines, (B) free body diagram of the unit cell of hierarchical honeycomb.