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Recursive higher-order constraints for linkages with lower kinematic pairs

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ABSTRACT

The geometric constraints imposed on the bodies of a linkage determine its function and mobility. They are the basis for any computational analysis of the kinematics and dynamics of a linkage, which ultimately requires higher-order time derivatives of the constraints. A tailored formulation for linkages comprising lower kinematic pairs is the product of exponentials (POE) formula, where rigid body motions are represented as curves in the Lie group $SE(3)$. The corresponding velocity constraints involve the instantaneous joint screws, and their derivatives involve the derivatives of these screws. It is known that partial derivatives of arbitrary order of the instantaneous joint screws are given explicitly and algebraically in terms of Lie brackets (i.e. screw products). This, however, leads to complex expressions that are very difficult to use in actual computations. In this paper a recursive formulation for the time derivatives of arbitrary order of the velocity constraints for lower-pair linkages is presented. This formulation applies to multi-loop linkages. To this end, the linkage topology is represented by a topological graph, and loop closure constraints are formulated for the (topologically independent) fundamental cycles. It is briefly discussed that this provides the basis for the higher-order kinematic analysis of linkages.

1. Introduction

This paper addresses the recursive formulation of higher-order loop closure constraints for mechanisms whose joints can be represented as a combination of lower pair joints. Such mechanisms are simply termed linkages [9]. Lower pairs are characterized by surface-to-surface contact. Moreover, a lower pair is the mechanical generator of a motion subgroup, i.e. a subgroup of the Lie group $SE(3)$ [29]. A lower pair joint with DOF $\delta = 1, 2, 3, 4, 6$ is in turn conveniently represented as an aggregate of δ screw joints, with revolute and prismatic joints as special cases. E.g. a spherical joint can be represented as combination of three revolute joints. The motion of these screw joints is then described in terms of screw coordinates and joint variables (angle, displacement). Mathematically this corresponds to a product decomposition of the joint's motion space into δ one-dimensional subgroups. The joint variables represent canonical coordinates of second kind on the motion subgroup. This is a well-accepted approach for the kinematic analysis of mechanisms [13,19,28], because: 1.) the instantaneous kinematics is described in terms of the linkage's screw system, 2.) the finite motion is determined by exponentials of the joint screws, and 3.) the derivatives of kinematic relations are given algebraically in terms of Lie brackets (screw products) of the joint screws. This allows, for instance, to:1.) invoke screw theory to identify and classify special configurations where screws become dependent [10,16], 2.) to obtain compact forward kinematics formulations using a single reference frame [1] (in contrast to DH parameters, for instance), and 3.) to derive coordinate invariant closed form expressions for velocity, acceleration, jerk, etc. [13,15,20,28] . Whereas the first two

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features are being widely exploited, the significance of the explicit formulation of higher-order constraints is not sufficiently recognized yet. Moreover, as shown in this paper, these yield computationally efficient recursive relations.

Using concepts from Lie group theory, the fact that the finite motion of a rigid body is a screw motion, has first been used in [7,8] to model constraints for single-loop linkages with a product of exponentials (POE) formula. The fact that the Lie group setting allows for coordinate free descriptions of the Euclidean motion, gave further rise to an abstract and coordinate free formulation of the kinematics and dynamics of linkages [3]. In the robotics context, the POE formula has originally been proposed in [1] for modeling the kinematics of serial manipulators. From a computational point of view, the main advantage of this formulation is its frame invariance and that partial derivatives of all kinematic entities are determined algebraically. This has been exploited for deriving the motion equations of multibody systems (MBS) [18,25,26], where second order derivatives are required. This has also been exploited for POE based methods for robot calibration [6,24]. Higher derivatives of the closure constraints are crucial for the higher-order kinematic analysis of linkages, and the explicit relations have been derived in closed form in terms of Lie brackets of the joint screws in [2,5,15,17,19,28], and used for the mobility analysis. It turned out, however, that the time derivatives of a screw system of degree higher than 4 are too complex to be derived manually. Recently the closed form of the partial derivatives of a linkage's screw system of arbitrary degree have been reported [20,21]. Therewith the time derivatives, and hence all higher-order constraints are available in principle. However, the complexity of the expressions complicates their actual computational evaluation.

To overcome this problem, the contribution of this paper is a recursive formulation of the constraints of arbitrary order (acceleration, jerk, jounce, etc.). This formulation only involves Lie brackets (screw products) of the instantaneous joint screws, which makes it computationally very efficient. It thus solves a long-standing problem in mechanism theory. The paper is organized as follows. In Section 2 the geometric constraints for a single kinematic loop are formulated using the POE formula in terms of joint screw coordinates. The corresponding velocity (i.e. first-order) constraints are derived in closed form in Section 3. The higher-order constraints are reported in Section 4. First the known closed form formulation are recalled, and their complexity is discussed. Then the novel recursive formulation is derived. This formulation is extended in Section 5 to multi-loop linkages. It is briefly outlined in Section 6 how this recursive formulation can be employed for the computational analysis (mobility determination and singularity analysis) of mechanisms.

2. Geometric constraints

Most technical joints can be represented as combination of lower-pair joints. The latter correspond to motion subgroups of $SE(3)$. The motion of lower-pairs can be (locally) represented as successive motion of 1-DOF screw joints. Consider first an open kinematic chain comprising n 1-DOF screw joints. The joints are indexed with $i = 1, \ldots, n$ starting from a reference body (e.g. the ground). The joint variables (angles, displacements) are denoted with q_i, and summarized in the vector $\mathbf{q} = (q_1, \ldots, q_n) \in \mathbb{V}^n$.

Let $\mathbf{e}_i$ be a unit vector along the screw axis of joint i, and $\mathbf{s}_i$ be the position vector to a point on that axis, both expressed in a world-fixed frame, and denote with h the pitch of the joint. Then $\mathbf{Y}_i = (\mathbf{e}_i, \mathbf{s}_i \times \mathbf{e}_i + h_i\mathbf{e}_i)^T$ is the screw coordinate vector [29] of joint i w.r.t. the world frame in the zero reference configuration $\mathbf{q} = 0$. The relative screw motion due to the 1-DOF joint i is given by the exponential mapping on $SE(3)$ as $\exp(\mathbf{Y}_i q_i)$ (see (28) in appendix). Combination of all n joint motions gives rise to the POE formula [1,29]

$$f(\mathbf{q}) = \exp(\mathbf{Y}_1 q_1)\exp(\mathbf{Y}_2 q_2)\cdot \ldots \cdot \exp(\mathbf{Y}_n q_n). \tag{1}$$

The constraint mapping $f : \mathbb{V}^n \to SE(3)$ can be considered to determine the motion of the terminal body in the kinematic chain, comprising n joints, w.r.t. the global frame. If the kinematic chain forms a closed loop, then joint 1 and n connect to the same body. That is, the terminal body of the chain is immobile, which is expressed by the loop closure condition

$$f(\mathbf{q}) = \mathbf{I}. \tag{2}$$

These are the geometric constraints for a kinematic loop comprising lower pair joints, and f in (1) is called the geometric *constraint mapping* of the kinematic loop. This formulation is computationally simple as it only requires the joint screw coordinates in the reference configuration.

It should be remarked that this formulation is also known as cut-body approach. This stems from the interpretation that the body linking joint n and joint 1 is cut into two halves, and the loop closure ensures the assembly of this body.

Remark 1 (*Frame invariance*). An important aspect of formulating constraints on $SE(3)$ using the POE is the inherent frame-invariance. Let $\mathbf{M} \in SE(3)$ be the transformation to another global reference frame. Then the constraints (2) transform according to $\bar{f}(\mathbf{q}) := \mathbf{M}f(\mathbf{q})\mathbf{M}^{-1} = \mathbf{I}$. The constraint mapping $\bar{f}(\mathbf{q}) = \exp(\overline{\mathbf{Y}}_1 q_1)\cdot \ldots \cdot \exp(\overline{\mathbf{Y}}_n q_n)$ is given in terms of the joint screw coordinates $\overline{\mathbf{Y}}_i = \mathbf{Ad}_{\mathbf{M}}\mathbf{Y}_i$ in the new reference frame (Appendix A). As a consequence all statements and results can be expressed