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Restrictions of point estimate methods and remedy

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ABSTRACT

Point estimate methods (PEMs) are widely applied to approximately estimate the moments and uncertainty of a random output variable in reliability evaluation and other engineering applications due to its computational efficiency and satisfactory accuracy. However, as any numerical approximation method, PEMs have their own limitations and should be used with caution. Unfortunately, although a lot of papers on point estimate methods have been published so far, almost none of them mentioned the limitations and restrictions of PEMs. In this paper, the restrictions of PEMs are theoretically analyzed and discussed with illustrative examples. A remedy for the restrictions is presented.

1. Introduction

Uncertainty exists inherently in reliability evaluation, risk assessment and many other engineering applications due to randomness of variables, inaccuracy of models, or lack of knowledge of objects under study [1–3]. To make a better decision in a reliability strategy or an actual application, the uncertainty of the output variable Z that is a function F of input random variables $X_1,X_2,\ldots,X_n$ needs to be quantified. In real life, the function could be very complicated and even may not have an analytical expression [1–3]. In this case, analytically calculating the uncertainty of the output variable from input variables is a challenge.

To tackle this issue, point estimate methods were introduced by Rosenblueth in his seminal papers for the first time [4,5] and then improved subsequently by other researchers [6–14]. The basic idea of PEMs to approximate the mean of the output variable Z is to compute a weighted sum of values of the function F evaluated in some deliberately chosen points. The choice of evaluation points is typically based on some statistical features of the input random variables, such as mean, variance, skewness, kurtosis, etc. Approximation of high-order raw moments works in a similar fashion. Once the raw moments of an output variable are estimated, the probability density function (PDF) of the output variable can be reconstructed approximately by employing Edgeworth expansion if necessary. Because of their high efficiency and low requirement for knowledge of input variables, PEMs have gained popularity in reliability evaluation and other engineering areas, and are widely applied to solve practical problems, such as applications in geotechnical engineering [15,16], electric power engineering [17–21], etc.

Although the aforementioned works [4–14] in PEMs share the same basic idea, they differ in the strategy of choosing evaluation points. In Rosenblueth's original point estimation scheme, the corner points of an n-dimensional hypercube around the mean vector of input variables are selected, which results in 2^n evaluations of the function, where n is the number of input variables. The disadvantage of this scheme is that when n is large, it becomes very inefficient or even infeasible. For example, for $n=20$, it requires more than one million of evaluations, which would be unacceptable in many actual applications. To improve the applicability of PEMs, instead of evaluating the function at each corner point, Lind [6] chose points near the center of each face of the hypercube. This gives a scheme with $n(n+1)/2$ points. Although the number of evaluation points is significantly reduced, it is still quadratic to the number of input variables. Schemes with the number of evaluation points with a linear relation to the number of input variables emerged since late 1980s. Harr [8] proposed a $2n$ scheme based on the eigen-decomposition of the correlation matrix of input variables. The requirement of correlation matrix restricts the applicability of the scheme since in many applications it is very difficult to obtain the correlation matrix with a reasonable precision. In contrast, the approach developed by Hong [11] is directly based on Taylor expansion and does not require any information of input variables beyond the first four statistical moments. Also, this approach takes into account the input variables' moments higher than the 2nd moment for the first time. Hong showed that this approach can make the estimation more accurate and at the same time uses only $2n+1$

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evaluation points. An excellent review of various popular point estimate schemes can be found in [22]. It should be pointed out that Hong [11] provided a firm derivation of PEMs based on Taylor series and other methods including Rosenblueth's original scheme can be derived from Hong's expression.

However, PEMs should be used with caution due to several inherent limitations. Unfortunately, researchers in the engineering area seem unaware of the limitations when applying PEMs to probabilistic engineering problems. Even developers of point estimate schemes did not mention assumptions and restrictions of PEMs. When they compared different point estimate schemes, their focus was placed on computational efficiency, empirical accuracy, treatment of bounded variables and correlations between variables [4–14,22].

In fact, PEMs do not directly work well for some types of functions. The objective of this paper is to present the restrictions of PEMs, discuss reasons, demonstrate illustrative examples, and propose a remedy for the restrictions. In practice, because of difficulty in obtaining accurate estimation of high order moments and consideration on computational burdens, a $2n$ scheme or $3n$ scheme, which can offers sufficient accuracy, is preferred. Since all PEMs are based on the same assumptions and have similar mathematical expressions and features, the $2n+1$ scheme is used as an example in the discussions of the paper.

2. Derivation of $2n+1$ PEM

It is necessary to briefly give a strict derivation process of PEMs so that the assumptions and features as sources of the restrictions can be manifested. In our view, the related papers except Hong's paper did not explicitly touch the mathematical essence of PEMs, which is the concept of Taylor series, although other PEMs implicitly used this concept as they can be re-derived from Taylor series based derivation. Using a similar idea in Hong's $2n+1$ scheme on a single variable case [11], the derivation of $2n+1$ scheme of PEMs on multiple variables is presented first as follows in order to demonstrate the assumptions and limitations of PEMs.

Let $J(\mathbf{x})$ be the joint distribution function of random variable vector $\mathbf{X}=(X_1,X_2,\ldots,X_n)$. The kth raw moment of $Z=F(\mathbf{X})$ is the expectation of Z^k with respect to$J(\mathbf{x})$

$$E\left(Z^k\right)\equiv\int_{\mathbf{D}}F^k(\mathbf{x})dJ(\mathbf{x}) \tag{1}$$

The integral is Remann–Stieltjes integral and $\mathbf{D}$ is the domain of $\mathbf{X}$. Mathematically, all of PEMs are to approximate this integral using a weighted sum of function values which are evaluated at a few selected points of input variable vector $\mathbf{X}$. They differ in the way of how to select the points and weighting factors.

Assuming $X_1,X_2,\ldots,X_n$ are statistically independent and expanding $Z=F(\mathbf{X})$ by Taylor series at the point $(\mu_1,\mu_2,\ldots,\mu_n)$ where $\mu_t(t=1,2,\ldots,n)$ is the mean of variable X_t, we have

$$Z=\sum_{m_1=0}^{\infty}\cdots\sum_{m_n=0}^{\infty}\frac{(x_1-\mu_1)^{m_1}\cdots(x_n-\mu_n)^{m_n}}{m_1!\cdots m_n!}\left(\frac{\partial^{m_1+\cdots+m_n}F}{\partial x_1^{m_1}\cdots\partial x_n^{m_n}}\right)(\mu_1,\ldots,\mu_n) \tag{2}$$

Note that the term $(\partial^{m_1+\cdots+m_n}F/\partial x_1^{m_1}\cdots\partial x_n^{m_n})(\mu_1,\ldots,\mu_n)$ in the equation represents the partial derivative $(\partial^{m_1+\cdots+m_n}F(x_1,x_2,\ldots,x_n)/\partial x_1^{m_1}\cdots\partial x_n^{m_n})$ at the point $(\mu_1,\ldots,\mu_n)$. If the series above converges to $F(\mathbf{X})$ for each value of $\mathbf{X}$, the mean μ_Z of Z can be expressed by

$$\mu_Z=E(F(\mathbf{X}))=\int_{\mathbf{D}}F(\mathbf{X})dJ(\mathbf{X}) \tag{3}$$

$$=\sum_{m_1=0}^{\infty}\cdots\sum_{m_n=0}^{\infty}\frac{\lambda_{1,m_1}\sigma_1^{m_1}\cdots\lambda_{n,m_n}\sigma_n^{m_n}}{m_1!\cdots m_n!}\left(\frac{\partial^{m_1+\cdots+m_n}F}{\partial x_1^{m_1}\cdots\partial x_n^{m_n}}\right)(\mu_1,\ldots,\mu_n) \tag{4}$$

where σ_t is the standard deviation of X_t and $\lambda_{t,j}\equiv\left(E(X_t-\mu_t)^j\right)/\sigma_t^j$.

Let function $h_t(x)$ of X_t be $F(\mu_1,\ldots,\mu_{t-1},x,\mu_{t+1},\ldots,\mu_n)$ which means that only X_t is the changeable variable while the values of other variables are fixed at $\mu_i(i\neq t)$. Applying the concept of Taylor series again, we obtain

$$h_t(x)\equiv h_t(\mu_t)+\sum_{i=1}^{\infty}\frac{1}{i!}h_t^{(i)}(\mu_t)(x-\mu_t)^i \tag{5}$$

where $h_t^{(i)}=\left(\partial^iF/\partial x_t^i\right)$. Let $x_{t,k}=\mu_t+\sigma_t\xi_{t,k},(k=1,2,3)$ where $\xi_{t,1}$ and $\xi_{t,2}$ are the constants to be determined and $\xi_{t,3}$ is set to zero. A direct result is $x_{t,3}=\mu_t$. Let $\omega_{t,k}$ be the weighting factor at point $(\mu_1,\mu_2,\ldots,x_{t,k},\ldots,\mu_n)$. Define

$$S\equiv\sum_{t=1}^{n}\left(\omega_{t,1}h_t(x_{t,1})+\omega_{t,2}h_t(x_{t,2})+\omega_{t,3}h_t(\mu_t)\right) \tag{6}$$

$$=F(\mu_1,\mu_2,\ldots,\mu_n)\sum_{t=1}^{n}\omega_{t,3}+\sum_{i=1}^{\infty}\sum_{t=1}^{n}\frac{1}{i!}h_t^{(i)}(\mu_t)\left(\omega_{t,1}\xi_{t,1}^i+\omega_{t,2}\xi_{t,2}^i\right)\sigma_t^i \tag{7}$$

It is noted that the series of S and the series of μ_Z are in such a similar form that each term containing the function or the product of partial derivative and standard deviation is corresponding to each other in the two series. This enables us to approximate μ_Z using S by matching up the first several terms (upto a degree of 4) of the two series. This is achieved by setting

$$\sum_{t=1}^{n}\left(\omega_{t,1}+\omega_{t,2}+\omega_{t,3}\right)=1 \tag{8}$$

$$\omega_{t,1}\xi_{t,1}^i+\omega_{t,2}\xi_{t,2}^i=\lambda_{t,i},\quad i=1,2,3,4,\quad t=1,2,\ldots,n \tag{9}$$

It can be assumed that the weighting factors are evenly distributed among all the variables X_t [11], namely,

$$\omega_{t,1}+\omega_{t,2}+\omega_{t,3}=\frac{1}{n},\quad t=1,2,\ldots,n \tag{10}$$

By solving the $5n$ simultaneous Eqs. in (9) and (10) for random variable X_t $(t=1,2,\ldots,n)$, we obtain the following standard locations for each variable

$$\xi_{t,k}=\begin{cases}\frac{\lambda_{t,3}}{2}+(-1)^{3-k}\sqrt{\lambda_{t,4}-\frac{3}{4}\lambda_{t,3}^2}, & k=1,2\\ 0, & k=3\end{cases} \tag{11}$$

and their corresponding weighting factors

$$\omega_{t,k}=\begin{cases}(-1)^{3-k}\dfrac{1}{\xi_{t,k}\left(\xi_{t,1}-\xi_{t,2}\right)} & k=1,2\\ \dfrac{1}{n}-\dfrac{1}{\lambda_{t,4}-\lambda_{t,3}^2} & k=3\end{cases} \tag{12}$$

By defining

$$\omega_0\equiv\sum_{t=1}^{n}\omega_{t,3}=1-\sum_{t=1}^{n}\frac{1}{\lambda_{t,4}-\lambda_{t,3}^2} \tag{13}$$

it follows that

$$\mu_Z\approx S=\omega_0F(\mu_1,\mu_2,\ldots,\mu_n)+\sum_{t=1}^{n}\sum_{i=1}^{2}\omega_{t,i}F(\mu_1,\ldots,\mu_{t-1},x_{t,i},\mu_{t+1},\ldots,\mu_n) \tag{14}$$