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Asymptotics for continuous lifetime distributions with polynomial failure rate with an application in reliability

Attila Csenki *

School of Computing, Informatics and Media, University of Bradford, Bradford, West Yorkshire BD7 1DP, UK

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Dedicated to Professor Hermann Witting (1927–2010) who taught me Probability Theory and Statistics at Freiburg University in the 1970s. He will always be remembered for his lucid and fascinating lectures.

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ABSTRACT

This paper is a continuation of work reported by Csenki (2011 [1]), where the coefficients of a polynomial failure rate function of a continuous lifetime distribution were expressed in terms of derivatives of the Laplace transform of the distribution at the origin. Here it is shown that these expressions are asymptotically normal. The main tool employed is the Delta Method in conjunction with the Central Limit Theorem. The finding is used to derive asymptotic confidence intervals and tests for the coefficients. The suggested calculations are carried out for a set of bus failure data from the literature.

1. Introduction

This paper is a continuation of work carried out in [1]. It was shown there that the coefficients of the failure rate function of a continuous lifetime distribution with polynomial failure rate can be expressed in terms of derivatives of the Laplace transform of the distribution at the origin. We are going to examine here the asymptotic behaviour of these expression (and the consequences thereof) by applying the *Delta Method* in conjunction with the *Central Limit Theorem*.

Section 2 contains a summary of the results from [1], leading in Section 3 to limit theorems: first, via the Central Limit Theorem to the asymptotic normality of sample moments, and then by the Delta Method to the asymptotic normality of the sample polynomial coefficients. Asymptotic tests and confidence intervals are derived in Section 4 from these limit results. The special case of a quadratic failure rate function is considered in more detail in Section 5. The theoretical results are applied in Section 6 to a failure data set from [4] that is known to come from a distribution with a quadratic failure rate function. The analysis shows that the hypothesis that the failure rate function is a first order polynomial must be rejected. We finish in Section 7 with a conclusion and an indication of some future research.

2. Background, notation and earlier results

2.1. Framework and notation

The framework is identical to that in [1] and we start by summarizing it here.

T models the time to failure of a system with a continuous distribution on $(0,\infty)$. The *failure rate* $r(t)$ is the conditional probability of failure in the next infinitesimal time unit given survival until time t; it is defined by

$$r(t)=\lim_{\Delta t\to 0}\frac{P(T\in(t,t+\Delta t)\,|\,T>t)}{\Delta t}.$$

If f and F denote respectively the probability density function (pdf) and the cumulative distribution function of T, then, it is well known that

$$r(t)=\frac{f(t)}{1-F(t)}. \tag{1}$$

r is assumed to be a *polynomial* of degree $m\in\mathbb{N}$ with

$$r(t)=a_0+a_1t+a_2t^2+\cdots+a_mt^m.$$

* Tel.: +44 1274 234277.
E-mail address: a.csenki@bradford.ac.uk

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The symbol $\xrightarrow{\mathcal{D}}$ will be used to denote convergence in distribution, $\mathcal{N}$ stands for a normal distribution (of appropriate dimension) and Φ is the standard normal distribution function. The transpose of a matrix (or vector) will be denoted by a superscript t.

2.2. Earlier results

A differential equation was derived in [1] for

$$\hat{f}(\tau) = \int_0^{+\infty} e^{-t\tau} f(t)\,dt,$$

the Laplace Transform (LT) of T; it is as follows:

$$\hat{f}(\tau) = \sum_{k=0}^{m} (-1)^k a_k \frac{d^k}{d\tau^k}\left(\frac{1}{\tau}(1-\hat{f}(\tau))\right), \quad \tau > 0. \tag{2}$$

Based on (2), an algorithm was described in [1] for writing the $m+1$ polynomial coefficients $a_0,\ldots,a_m$ as functions of finitely many of the derivatives of the LT $\hat{f}$ at the origin:

$$a_i = a_i(\hat{f}_1,\ldots,\hat{f}_\kappa), \quad i=0,\ldots,m, \tag{3}$$

where

$$\hat{f}_i = \frac{d^i}{d\tau^i}(\hat{f}(\tau))\bigg|_{\tau\to 0} = (-1)^i E(T^i). \tag{4}$$

A process for finding the index κ in (3) and the functional relationships themselves is described in [1]. Eq. (4) shows that the quantities $\hat{f}_1,\ldots,\hat{f}_\kappa$ can be estimated from sample moments of T. The computer algebra system Maxima [2,8,9] was used in [1] to work out special cases explicitly.

3. Limit theorems

Let $T_1,\ldots T_\ell$ be ℓ i.i.d. random variables whose distribution is that of T. Then, the random variable

$$\hat{\mu}_{i,\ell} = \frac{1}{\ell}\sum_{j=1}^{\ell}(-T_j)^i \tag{5}$$

is by (4) a consistent and unbiased estimator of $\hat{f}_i$ as $\ell\to\infty$. Furthermore, by the multivariate *Central Limit Theorem* (e.g. [5, p. 313]) we have for $\ell\to\infty$,

$$\sqrt{\ell}\begin{pmatrix} \hat{\mu}_{1,\ell}-E(-T) \\ \hat{\mu}_{2,\ell}-E((-T)^2) \\ \vdots \\ \hat{\mu}_{\kappa,\ell}-E((-T)^\kappa) \end{pmatrix} \xrightarrow{\mathcal{D}} \mathcal{N}(\mathbf{0},\mathbf{C}), \tag{6}$$

where the ijth entry of the $\kappa\times\kappa$ covariance matrix $\mathbf{C}$ is given by

$$\begin{aligned} c_{ij} &= Cov((-T)^i,(-T)^j) = (-1)^{i+j}Cov(T^i,T^j) \\ &= (-1)^{i+j}(E(T^{i+j})-E(T^i)E(T^j)) = \hat{f}_{i+j}-\hat{f}_i\hat{f}_j. \end{aligned} \tag{7}$$

By the multivariate version of the *Delta Method* (e.g. [5, p. 315]), (6) implies for $\ell\to\infty$ that

$$\sqrt{\ell}\begin{pmatrix} a_0(\hat{\mu}_{1,\ell},\ldots,\hat{\mu}_{\kappa,\ell})-a_0(\hat{f}_1,\ldots\hat{f}_\kappa) \\ \vdots \\ a_m(\hat{\mu}_{1,\ell},\ldots,\hat{\mu}_{\kappa,\ell})-a_m(\hat{f}_1,\ldots\hat{f}_\kappa) \end{pmatrix} \xrightarrow{\mathcal{D}} \mathcal{N}(\mathbf{0},\mathbf{\Delta}), \tag{8}$$

where

$$\mathbf{\Delta} = (\delta_{\mu\nu} : \mu,\nu = 1,\ldots,m+1) = \mathbf{JCJ}^t, \tag{9}$$

and $\mathbf{J}$ in (9) denotes the Jacobian

$$\mathbf{J}(x_1,\ldots,x_\kappa) = \begin{pmatrix} \frac{\partial a_0(x_1,\ldots,x_\kappa)}{\partial x_2} & \cdots & \frac{\partial a_0(x_1,\ldots,x_\kappa)}{\partial x_\kappa} \\ \vdots & \ddots & \vdots \\ \frac{\partial a_m(x_1,\ldots,x_\kappa)}{\partial x_2} & \cdots & \frac{\partial a_m(x_1,\ldots,x_\kappa)}{\partial x_\kappa} \end{pmatrix} \tag{10}$$

of the mapping

$(a_0,\ldots,a_m)^t : \mathbb{R}^\kappa \to \mathbb{R}^{m+1}$

in (3), *evaluated at* $(\hat{f}_1,\ldots\hat{f}_\kappa)$. ($\mathbf{J}$ is of size $(m+1)\times\kappa$.) The entries of the $(m+1)\times(m+1)$ covariance matrix $\mathbf{\Delta}$ in (9) are

$\delta_{\mu\nu} = \delta_{\mu\nu}(\hat{f}_1,\ldots\hat{f}_{2\kappa}),$

where by (7) and (10) it is

$$\delta_{\mu\nu}(x_1,\ldots,x_{2\kappa}) = \sum_{i=1}^{\kappa}\sum_{j=1}^{\kappa}(x_{i+j}-x_ix_j)\frac{\partial a_{\mu-1}(x_1,\ldots,x_\kappa)}{\partial x_i}\frac{\partial a_{\nu-1}(x_1,\ldots,x_\kappa)}{\partial x_j}. \tag{11}$$

We mention in passing that the Delta Method is a well known and much used asymptotic tool, as seen, for example, in the recent papers [6,10].

4. Tests and confidence intervals

A pertinent question is whether an assumed polynomial failure rate function of a certain degree m can be replaced by a simpler one of degree $m-1$. Below an asymptotic test and a confidence interval are developed for this problem.

It follows from (8) that $a_m(\hat{\mu}_{1,\ell},\ldots,\hat{\mu}_{\kappa,\ell})$ is asymptotically normal, more precisely, it is for $\ell\to\infty$:

$$\sqrt{\ell}(a_m(\hat{\mu}_{1,\ell},\ldots,\hat{\mu}_{\kappa,\ell})-a_m(\hat{f}_1,\ldots\hat{f}_\kappa)) \xrightarrow{\mathcal{D}} \mathcal{N}(0,\delta_{m+1m+1}(\hat{f}_1,\ldots\hat{f}_{2\kappa})),$$

which then by *Slutsky's Theorem* (e.g. [5]) and by continuity of δ_{m+1m+1} implies for $\ell\to\infty$ that

$$\sqrt{\ell}\frac{a_m(\hat{\mu}_{1,\ell},\ldots,\hat{\mu}_{\kappa,\ell})-a_m(\hat{f}_1,\ldots\hat{f}_\kappa)}{\sqrt{\delta_{m+1m+1}(\hat{\mu}_{1,\ell},\ldots,\hat{\mu}_{2\kappa,\ell})}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1). \tag{12}$$

Eq. (12) suggests an asymptotic rejection region R for a two-sided test of the null hypothesis $H_0 : a_m = 0$ to significance level $\alpha\in(0,1)$ where

$$R = \left\{(T_1,\ldots,T_\ell)\in\mathbb{R}^\ell : \left|\sqrt{\ell}\frac{a_m(\hat{\mu}_{1,\ell},\ldots,\hat{\mu}_{\kappa,\ell})}{\sqrt{\delta_{m+1m+1}(\hat{\mu}_{1,\ell},\ldots,\hat{\mu}_{2\kappa,\ell})}}\right| > u_{\alpha/2}\right\}, \tag{13}$$

and u_α is the $(1-\alpha)$-quantile of the standard normal distribution, i.e. $\Phi(u_\alpha) = 1-\alpha$.

The corresponding level-$(1-\alpha)$ asymptotic confidence interval for a_m is given by

$$a_m(\hat{\mu}_{1,\ell},\ldots,\hat{\mu}_{\kappa,\ell}) \pm \sqrt{\frac{\delta_{m+1m+1}(\hat{\mu}_{1,\ell},\ldots,\hat{\mu}_{2\kappa,\ell})}{\ell}}u_{\alpha/2}. \tag{14}$$

Similar results for *vectors* of the polynomial coefficients can be deduced from (8) in like manner. The confidence regions obtained thereby in the two dimensional case will be ellipses, whereas in general ellipsoids will be obtained.

5. Special case: quadratic failure rate

As the data set that will be analysed in Section 6 is from a distribution with a quadratic failure rate ($m=2$), this special case will now be considered in more detail.