Mechanism and Machine Theory 46 (2011) 1193–1206

Contents lists available at ScienceDirect

Mechanism and Machine Theory

journal homepage: www.elsevier.com/locate/mechmt

Mechanism mobility and a local dimension test☆

Charles W. Wampler [a,*,1], Jonathan D. Hauenstein [b,2], Andrew J. Sommese [c,3]

[a] General Motors Research and Development, Mail Code 480-106-359, 30500 Mound Road, Warren, MI 48090–9055, USA
[b] Department of Mathematics, Texas A&M University, Mailstop 3368, College Station, TX 77843–3368, USA
[c] Department of Applied & Computational Mathematics & Statistics, University of Notre Dame, Notre Dame, IN 46556–4618, USA

ARTICLE INFO

Article history:
Received 28 May 2010
Received in revised form 11 April 2011
Accepted 21 April 2011
Available online 25 May 2011

Keywords:
Mobility
Local dimension
Parallel manipulators
Overconstrained mechanisms
Self-motion
Numerical algebraic geometry

ABSTRACT

The mobility of a mechanism is the number of degrees of freedom (DOF) with which it may move. This notion is mathematically equivalent to the dimension of the solution set of the kinematic loop equations for the mechanism. It is well known that the classical Grübler–Kutzbach formulas for mobility can be wrong for special classes of mechanisms, and even more refined treatments based on displacement groups fail to correctly predict the mobility of so-called "paradoxical" mechanisms. This article discusses how recent results from numerical algebraic geometry can be applied to the question of mechanism mobility. In particular, given an assembly configuration of a mechanism and its loop equations, a local dimension test places bounds on the mobility of the associated assembly mode. A publicly available software code makes the idea easy to apply in the kinematics domain.

1. Introduction

The most basic property of a mechanism is its *mobility*, that is, its number of degrees of freedom (DOFs). One may also speak of the mobility of a family of mechanisms: for example, one may say that planar four-bar linkages have mobility one. Such statements are properly understood to mean that almost all of the mechanisms in the family have the stated mobility, although there may be exceptions. A classical example is the family of 4R spatial single-loop mechanisms. Four general R–R links cannot even be assembled into a closed loop, yet planar, spherical, and Bennett four-bars all assemble with mobility one. (Delassus [7] showed that there are no other moveable four-bars.) Another notable example is the family of Stewart–Gough parallel-link (6SPU) robots, which when the leg lengths are held fixed may be considered as 6SU mechanisms. Most 6SU mechanisms are *structures*, that is, they have mobility zero. They can be assembled in a finite number of configurations (at most 40) and are immobile in each of these. Nevertheless, exceptional cases exist of 6SU mechanisms that have mobility one, specifically the architecturally singular Stewart–Gough platforms as classified by Karger [15] and a moveable platform found by Geiss and Schreyer [8] that is not architecturally singular. The mobilities of many families of mechanisms, particularly those described by just a list of the number of links and the kind of joints between them, submit to simple formulas, such as the Grübler–Kutzbach formulas. In contrast, those families whose description includes extra geometric constraints, such as parallel or perpendicular joint axes or particular combinations of link lengths, often require a more detailed analysis.

☆ This material is based on work supported by the National Science Foundation under Grant No. 0712910.
* Corresponding author.
E-mail addresses: charles.w.wampler@gm.com (C.W. Wampler), jhauenst@math.tamu.edu (J.D. Hauenstein), sommese@nd.edu (A.J. Sommese).
URLs: http://www.nd.edu/~cwample1 (C.W. Wampler), http://www.math.tamu.edu/~jhauenst (J.D. Hauenstein), http://www.nd.edu/~sommese (A.J. Sommese).
1 Also supported by General Motors Research and Development.
2 Also supported by the Fields Institute.
3 Also supported by the Duncan Chair of the University of Notre Dame.

0094-114X/$ – see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.mechmachtheory.2011.04.011

The situation is even more complicated than just indicated, for some mechanisms have assembly modes of different mobility. The existence of such mechanisms forces one to speak of the mobility of each assembly mode rather than the mobility of the mechanism. It may even happen that two assembly modes of different mobility happen to meet, in which case the mobility of the mechanism can change at a point of intersection. Mechanisms with this property are said to be *kinematotropic* [26].

Kinematicians also speak of “finite mobility” and “infinitesimal mobility.” An infinitesimal degree of freedom corresponds to a direction of motion that exists to first or higher differential order but does not extend to a finite motion. As we shall discuss, these degrees of freedom are intimately connected to the concept of roots that have multiplicity greater than one. All degrees of freedom, finite and infinitesimal, lie in the null-space of the Jacobian matrix for the loop equations. This article describes an extension of the Jacobian matrix, called a Macaulay matrix, which includes higher order terms that can be used to distinguish between finite and infinitesimal directions, thus arriving at the finite mobility of the mechanism. The methodology involved comes from work in numerical algebraic geometry, where the Macaulay matrix is central to a *local dimension test* that is used to sort solution points found by numerical continuation [1].

In short, the contribution of this article is to show how the local dimension test from numerical algebraic geometry applies to determining the mobility of an assembly mode of a mechanism. We describe an algorithm whose inputs are: (1) a mechanism as defined by its loop equations, (2) an assembly configuration of the mechanism, and (3) an upper limit on the order of the analysis. The output is a determination of the local mobility up to the given order. We show how the method can often be applied to a whole mechanism family. Care is taken to clarify the mathematical meaning of the computed results.

The paper begins with a short review of mobility analysis as currently conducted in the kinematics community. We then review Macaulay matrices and the local dimension test based on them. This leads to a new approach to computing mobility, which we illustrate on several examples.

2. Mobility analysis

To place the current work in context, we begin with a brief review of existing methods for determining the mobility of mechanisms and mechanism families. A more detailed review of the field is available in [9].

The idea that underlies formulas for calculating mobility is basically a count of the number of variables and the number of constraint equations, the latter being the loop closure equations for a mechanism. The difference between these is a first guess at the mobility of the mechanism, as each (scalar) loop equation has the potential to reduce the mobility by one. However, this guess is only correct if each of the equations places an independent constraint on the motion. The question of independence is thus at the heart of the matter.

The need for a more refined approach than counting variables and equations is immediately apparent in the kinematics context. Consider a rigid body in three-space. Its location is described by a position and orientation, say $(p,R)\in SE(3) = \mathbb{R}^3 \times SO(3)$. Here, $SO(3)$ is the set of 3×3 rotation matrices given by

$$SO(3) = \left\{ R \in \mathbb{R}^{3\times 3} \middle| R^T R = I, \qquad \left|R\right| = 1 \right\}. \tag{1}$$

It is well-known that $\dim SO(3) = 3$, but this is not immediately apparent from a count of variables and equations, as detailed next.

Example 2.1 (SO(3)). Matrix R in Eq. (1) has nine entries. Due to symmetry, the matrix equation $R^T R = I$ is equivalent to just six scalar equations, so with the final equation $|R| = 1$, there are a total of seven. If the seven equations were independent, one would have that $SO(3)$ is two dimensional $(9-7=2)$, whereas it is known to be three dimensional. The first six equations determine two sets of dimension three, the set of rotations having $|R| = 1$ and the set of mirror-image rotations having $|R| = -1$. Thus the final equation, $|R| = 1$, does not reduce the dimension of the set; instead, it picks out the rotations and discards their mirror images.

The simplest mobility formulas, which we refer to as Grübler–Kutzbach formulas [10,16], account for the dimension of the ambient motion space of the links. A free-floating rigid body in three-space has six degrees of freedom, the dimension of $SE(3)$. Thus, $\mathcal{D} = 6$ is the ambient dimension for spatial mechanisms. But for planar or spherical mechanisms, the ambient dimension is $\mathcal{D} = 3$ being the dimension of $\mathbb{R}^2 \times SO(2)$ and $SO(3)$, respectively. Declaring one link as a fixed ground link, we have that a mechanism built with N links has $\mathcal{D}(N-1)$ degrees of freedom before connecting the links with joints. Suppose that a two-link mechanism with a single joint of a certain type (e.g., revolute or prismatic) has $\mathcal{F}$ degrees of freedom. This implies that the joint removes $\mathcal{C} = \mathcal{D} - \mathcal{F}$ degrees of freedom, and $\mathcal{C}$ is called the degree of constraint of the joint. The Grübler–Kutzbach formulas assume that such a joint always removes $\mathcal{C}$ freedoms no matter where it is placed in a multi-link mechanism so that the resulting mobility is:

$$DOF = \mathcal{D}(N-1) - \sum_{j\in J} \mathcal{C}_j = \sum_{j\in J} \mathcal{F}_j - \mathcal{D}\mathcal{L}, \tag{2}$$

where J is the set of joints, $\mathcal{C}_j$ is the number of constraints imposed by joint j, $\mathcal{F}_j$ is the number of freedoms allowed by joint j, and $\mathcal{L}$ is the number of loop closures. Variants of the formulas derive from topological relations between the number of links, joints, and loops. The degrees of freedom of the basic “lower-order pair” joints and their kinematic symbols are given in Table 1.