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Sensitivity analysis in optimization and reliability problems

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ABSTRACT

The paper starts giving the main results that allow a sensitivity analysis to be performed in a general optimization problem, including sensitivities of the objective function, the primal and the dual variables with respect to data. In particular, general results are given for non-linear programming, and closed formulas for linear programming problems are supplied. Next, the methods are applied to a collection of civil engineering reliability problems, which includes a bridge crane, a retaining wall and a composite breakwater. Finally, the sensitivity analysis formulas are extended to calculus of variations problems and a slope stability problem is used to illustrate the methods.

1. Introduction

This paper deals with sensitivity analysis. Sensitivity analysis discusses "how" and "how much" changes in the parameters of an optimization problem modify the optimal objective function value and the point where the optimum is attained (see [1]).

Today, it is not enough to give users the solutions to their problems. In addition, they require knowledge of how these solutions depend on data and/or assumptions. Therefore, data analysts must be able to supply the sensitivity of their conclusions to model and data. Sensitivity analysis allows the analyst to assess the effects of changes in the data values, to detect outliers or wrong data, to define testing strategies, to increase the reliability, to optimize resources, reduce costs, etc.

Sensitivity analysis increases the confidence in the model and its predictions, by providing an understanding of how the model responds to changes in the inputs. Adding a sensitivity analysis to a study means adding extra quality to it.

Sensitivity analysis is not a standard procedure, however, it is very useful to (a) the designer, who can know which data values are the most influential on the design, (b) to the builder, who can know how changes in the material properties or the prices influence the total reliability or the cost of the work being designed, and (c) to the code maker, who can know the costs and reliability implications associated with changes in the safety factors or failure probabilities. The methodology proposed below is very simple, efficient and allows all the sensitivities to be calculated simultaneously. At the same time it is the natural way of evaluating sensitivities when optimization procedures are present.

The paper is structured as follows. In Section 2 the statement of optimization problems and the conditions to be satisfied are presented. Section 3 gives the formula to get sentivities with respect to the objective function. In Section 4, a general method for deriving all possible sensitivities is given. Section 5 deals with some examples and the interpretation of the sensitivity results. In Section 6 the methodology is extended to calculus of variations, and finally, Section 7 provides some relevant conclusions.

2. Statement of the problem

Consider the following primal non-linear programming problem (NLPP):

$$\underset{\boldsymbol{x}}{\text{Minimize}}\ z_P = f(\boldsymbol{x}, \boldsymbol{a}) \tag{1}$$

subject to

$$\boldsymbol{h}(\boldsymbol{x}, \boldsymbol{a}) = \boldsymbol{b} : \boldsymbol{\lambda}, \tag{2}$$

$$\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{a}) \leqslant \boldsymbol{c} : \boldsymbol{\mu}, \tag{3}$$

where $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$, $\boldsymbol{h}: \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^\ell$, $\boldsymbol{g}: \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^m$ with $\boldsymbol{h}(\boldsymbol{x}, \boldsymbol{a}) = (h_1(\boldsymbol{x}, \boldsymbol{a}), \ldots, h_\ell(\boldsymbol{x}, \boldsymbol{a}))^{\mathrm{T}}$, $\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{a}) = (g_1(\boldsymbol{x}, \boldsymbol{a}), \ldots, g_m(\boldsymbol{x}, \boldsymbol{a}))^{\mathrm{T}}$ are regular enough for the mathematical developments to be valid over the feasible region $S(\boldsymbol{a}) = \{\boldsymbol{x} | \boldsymbol{h}(\boldsymbol{x}, \boldsymbol{a}) = \boldsymbol{b}, \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{a}) \leqslant \boldsymbol{c}\}$, $f, \boldsymbol{h}, \boldsymbol{g} \in C^2$,

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and $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are the vectors of dual variables. It is also assumed that the problem (1)–(3) has an optimum at $\boldsymbol{x}^*$.

Any primal problem P, as that stated in (1)–(3), has an associated dual problem D, which is defined as

$$\underset{\boldsymbol{\lambda},\boldsymbol{\mu}}{\text{Maximize}}\ z_D = \operatorname*{Inf}_{\boldsymbol{x}} \mathscr{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) \tag{4}$$

subject to

$$\boldsymbol{\mu} \geqslant \mathbf{0}, \tag{5}$$

where

$$\mathscr{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) = f(\boldsymbol{x}, \boldsymbol{a}) + \boldsymbol{\lambda}^{\mathrm{T}}(\boldsymbol{h}(\boldsymbol{x}, \boldsymbol{a}) - \boldsymbol{b}) + \boldsymbol{\mu}^{\mathrm{T}}(\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{a}) - \boldsymbol{c}) \tag{6}$$

is the Lagrangian function associated with the primal problem (1)–(3), and $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, the dual variables, are vectors of dimensions ℓ and m, respectively. Note that only the dual variables ($\boldsymbol{\mu}$ in this case) associated with the inequality constraints ($\boldsymbol{g}(\boldsymbol{x})$ in this case), must be non-negative.

Given some regularity conditions on local convexity (see [2,3]), if the primal problem (1)–(3) has a locally optimal solution $\boldsymbol{x}^*$, the dual problem (4)–(6) also has a locally optimal solution $(\boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)$, and the optimal values of the objective functions of both problems coincide.

2.1. Karush–Kuhn–Tucker conditions

The primal (1)–(3) and the dual (4)–(6) problems, respectively, can be solved using the Karush–Kuhn–Tucker (KKTCs) first order necessary conditions (see, for example, [2,4,5]):

$$\nabla_x f(\boldsymbol{x}^*, \boldsymbol{a}) + \boldsymbol{\lambda}^{*\mathrm{T}} \nabla_x \boldsymbol{h}(\boldsymbol{x}^*, \boldsymbol{a}) + \boldsymbol{\mu}^{*\mathrm{T}} \nabla_x \boldsymbol{g}(\boldsymbol{x}^*, \boldsymbol{a}) = \mathbf{0}, \tag{7}$$

$$\boldsymbol{h}(\boldsymbol{x}^*, \boldsymbol{a}) = \boldsymbol{b}, \tag{8}$$

$$\boldsymbol{g}(\boldsymbol{x}^*, \boldsymbol{a}) \leqslant \boldsymbol{c}, \tag{9}$$

$$\boldsymbol{\mu}^{*\mathrm{T}}(\boldsymbol{g}(\boldsymbol{x}^*, \boldsymbol{a}) - \boldsymbol{c}) = 0, \tag{10}$$

$$\boldsymbol{\mu}^* \geqslant \mathbf{0}, \tag{11}$$

where $\boldsymbol{x}^*$ and $(\boldsymbol{\lambda}^*, \boldsymbol{\mu}^*)$ are the primal and dual optimal solutions, $\nabla_x f(\boldsymbol{x}^*; \boldsymbol{a})$ is the gradient (vector of partial derivatives) of $f(\boldsymbol{x}^*; \boldsymbol{a})$ with respect to $\boldsymbol{x}$, evaluated at the optimal value $\boldsymbol{x}^*$. The vectors $\boldsymbol{\mu}^*$ and $\boldsymbol{\lambda}^*$ are also called the *Kuhn–Tucker multipliers*. Condition (7) says that the gradient of the Lagrangian function in (6) evaluated at the optimal solution $\boldsymbol{x}^*$ must be zero. Conditions (8)–(9) are called *the primal feasibility* conditions. Condition (10) is known as the *complementary slackness condition*. Finally, condition (11) requires the non-negativity of the multipliers of the inequality constraints, and is referred to as the *dual feasibility conditions*.

Note that in the present analysis the regular non-degenerate case is only considered, which is the most frequent in real life applications. To understand the meaning of these assumptions the following definitions are given.

Definition 1 (*Regular point*). The solution $\boldsymbol{x}^*$ of the optimization problem (1)–(3) is said to be a regular point of the constraints if the gradient vectors of the active constraints are linearly independent.

Definition 2 (*Degenerate inequality constraint*). An inequality constraint is said to be degenerate if it is active and the associated μ-multiplier is null.

Once the optimal solution is known, degeneracy can be identified and possibly eliminated. The degenerate case is extensively analyzed in [3].

2.2. Some sensitivity analysis questions

When dealing with the optimization problem (1)–(3), the following questions regarding sensitivity analysis are of interest:

1. What are the local sensitivities of $z_P^* = f(\boldsymbol{x}^*, \boldsymbol{a})$ to changes in $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$?
2. What are the local sensitivities of $\boldsymbol{x}^*$ to changes in $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$?
3. What are the local sensitivities of $\boldsymbol{\lambda}^*$ and $\boldsymbol{\mu}^*$ to changes in $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$?

The answers to these questions are given in the following sections.

3. Sensitivities of the objective function

Calculating the sensitivities of the objective function with respect to data is extremely easy using the following theorem (see [6]).

Theorem 1. *Assume that the solution $\boldsymbol{x}^*$ of the above optimization problem is a regular point and that no degenerate inequality constraints exists. Then, the sensitivity of the objective function with respect to the parameter $\boldsymbol{a}$ is given by the gradient of the Lagrangian function*

$$\mathscr{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) = f(\boldsymbol{x}, \boldsymbol{a}) + \boldsymbol{\lambda}^{\mathrm{T}}(\boldsymbol{h}(\boldsymbol{x}, \boldsymbol{a}) - \boldsymbol{b}) + \boldsymbol{\mu}^{\mathrm{T}}(\boldsymbol{g}(\boldsymbol{x}, \boldsymbol{a}) - \boldsymbol{c}), \tag{12}$$

with respect to $\boldsymbol{a}$ evaluated at the optimal solution $\boldsymbol{x}^*$, $\boldsymbol{\lambda}^*$, and $\boldsymbol{\mu}^*$, i.e.

$$\frac{\partial z_P^*}{\partial \boldsymbol{a}} = \nabla_{\boldsymbol{a}} \mathscr{L}(\boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{\mu}^*, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}). \tag{13}$$

Note that this theorem is very useful from the practical point of view because it allows you to know how much the objective function value z_P^* changes when parameters $\boldsymbol{a}$ change.

Note that the sensitivities with respect to **b** and **c** are their respective gradients.

Example 1 (*Objective function sensitivity with respect to right-hand side parameters*). Consider the optimization problem (1)–(3). Using Theorem 1, i.e., differentiating (12) with respect to $\boldsymbol{b}$ and $\boldsymbol{c}$ one obtains:

$$\frac{\partial f(\boldsymbol{x}^*, \boldsymbol{a})}{\partial b_i} = -\lambda_i^*; \quad i = 1, 2, \ldots, \ell; \quad \frac{\partial f(\boldsymbol{x}^*, \boldsymbol{a})}{\partial c_j} = -\mu_j^*;$$
$$j = 1, 2, \ldots, m$$

i.e., the sensitivities of the optimal objective function value of the problem (1)–(3) with respect to changes in the terms appearing on the right-hand side of the constraints are the negative of the optimal values of the corresponding dual variables.

For this important result to be applicable to practical cases of sensitivity analysis, the parameters for which the sensitivities are sought must appear on the right-hand side of the primal problem constraints.

At this point the reader can ask him/herself and what about parameters not satisfying this condition, $\boldsymbol{a}$ for example? The answer to this question is given in the following example.

Example 2 (*A practical method for obtaining all the sensitivities of the objective function*). In this example it is shown how the duality methods can be applied to derive the objective function sensitivities in a straightforward manner. The basic idea is simple. Assume that we desire to know the sensitivity of the objective function to changes in some data values. If we convert the data into artificial variables and set them, by means of constraints, to their actual values, we obtain a problem that is equivalent to the