Mechanics Research Communications 70 (2015) 17–23

Contents lists available at ScienceDirect

Mechanics Research Communications

journal homepage: www.elsevier.com/locate/mechrescom

A model and numerical study for coiling of Kelvin-type viscoelastic filament

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ARTICLE INFO

Article history:
Received 9 January 2015
Received in revised form 27 August 2015
Accepted 2 September 2015
Available online 21 September 2015

Keywords:
Viscoelastic filament coiling
Kelvin model
Continuation method
Retardation time

ABSTRACT

Coiling problems of elastic rope and viscous jet have been fully studied both experimentally and theoretically, but very few studies exist for viscoelastic material. In this paper, a system of one-dimensional two-point boundary nonlinear equations for Kelvin material coiling is presented. The equations are solved numerically by continuation method. It is found that the coiling frequency depends on the dimensionless retardation time and other continuation parameters involving inertia and gravity. The multivaluedness of the solution is observed in the numerical study.

1. Introduction

The phenomenon of thin filament falling on the surface forming serious coiling circles aroused great interesting in recent decades. Coiling problems of purely elastic rope and viscous Newtonian jet have been studied in great detail. For purely elastic rope, Mahadevan and Keller [1] firstly proposed a numerical model based on Krichhoff type equations including the action of gravity and elastic force but with some sign problems of inertial term [2]. Recently, Habibi et al. [2] restudied this issue by laboratory experiments and gave out a correct version of Mahadevan's model, the numerical results of which are in excellent agreement with the experiment. Their researches show three basic modes appear for the elastic rope coiling: elastic, gravitational and inertial mode which involving different force balances of elastic force, gravity and inertia force. For viscous Newtonian jet, experimental and theoretical studies revealed three different modes including viscous, gravitational and inertial modes depending on the relative importance of viscous force, gravity and inertia force [3]. Also the instability and multistable behavior are observed both in the numerical study and experiments [4,5]. Ribe et al. [6] analyzed the stability of coiling problem through applying perturbations on time-dependent equations of thin viscous rope's motion. A recent review article [7] summarized the pioneer work in the field of viscous jet coiling and figured out that the exploration of ropes coiling problem for non-Newtonian rheology had only begun. Actually, there are some exploratory experimental researches of viscoelastic filament which have been carried out. Majmudar et al. [8] presented a systematic experimental study of the effects of viscoelasticity on the dynamics of liquid jet of surfactant solution, where the rheology of jet is fitted by Maxwell model. Rahmani et al. [9] presented an experimental investigation of coiling of two kinds of yield stress filaments, which were shaving foam behaving like solid and hair gel like liquid. Both working materials were measured by rheometer, with elastic and viscous modulus. These experiments illustrate the viscoelastic material coiling is different with elastic or viscous coiling. However, few theoretical and numerical studies exist for modeling the viscoelastic filament and analyzing its dynamics.

In this paper, we focus on the methodology of modeling and numerical studying of thin Kelvin-type viscoelastic filament coiling problem. Our goal is firstly to establish a system of one-dimensional governing equations for the steady state of viscoelastic filament coiling within the corotating reference frame that the filament is laid out in a circular coil with uniform radius. And the second goal is through numerical techniques such as continuation method and spectral method to find out the solutions of the problem in various parameter values, especially to check how the retardation time influences the behavior of coiling.

2. Governing equations for steady coiling of viscoelastic filament

2.1. Assumptions and conventions

Our analysis is based on several assumptions: (1) The filament is a naturally straight thin thread with circular cross-section. After

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http://dx.doi.org/10.1016/j.mechrescom.2015.09.001
0093-6413/

deformation, the central line becomes curve and is parameterized by the arc length s. The slenderness of filament is identify by the parameter $\varepsilon = d_0\kappa \ll 1$, where d_0 and κ are the diameter of the nozzle and the characteristic axial curvature respectively. (2) The diameter of the filament remains constant as a function of the arc length, which is equal to the diameter at the nozzle. The stretching of the filament due to gravity is neglected. This assumption corresponds to the observation of coiling of shave foam [9] and explained in the Section 5. (3) For thin filament, the shear deformation between cross sections is small compared to the bending and twisting deformation and neglected. Also shear deformation will not occur in the plane of cross section. (4) The material is nearly incompressible. Then the magnitude of velocity along the axis remains constant due to the conservation of mass.

There are two kinds of three-dimensional coordinate systems employed. One is local system defining the orientation of the cross section and the other is global frame to describe the gravity and central line position. For convenience, we use the generalized Einstein summation convention: Latin and Greek indices range over the values 1, 2, 3 and 1, 2 respectively. ($\mathbf{d}_1$,$\mathbf{d}_2$,$\mathbf{d}_3$) describes a local coordinate system, where $\mathbf{d}_1$ and $\mathbf{d}_2$ lie along the principal axes of the cross-section and $\mathbf{d}_3 = \mathbf{d}_1 \times \mathbf{d}_2$ which points along the axis of the rod. The local coordinate system is relative to global frame ($\mathbf{e}_1$,$\mathbf{e}_2$,$\mathbf{e}_3$) through a matrix $[\mathbf{L}] = [l_{ij}]$, which satisfies, $\mathbf{d}_i = l_{ij}\mathbf{e}_j$. [$\mathbf{L}$] is defined by a singularity-free parameterization in terms of the Euler parameters q_1, q_2, q_3 and q_0 as [1],

$$[\mathbf{L}] = \begin{bmatrix} q_1^2 - q_2^2 - q_3^2 + q_0^2 & 2(q_1q_2 + q_0q_3) & 2(q_1q_3 - q_0q_2) \\ 2(q_1q_2 - q_0q_3) & -q_1^2 + q_2^2 - q_3^2 + q_0^2 & 2(q_2q_3 + q_0q_1) \\ 2(q_1q_3 + q_0q_2) & 2(q_2q_3 - q_0q_1) & -q_1^2 - q_2^2 + q_3^2 + q_0^2 \end{bmatrix} \quad (1)$$

By default, all the components of vectors and tensors are expressed in the local coordinates below.

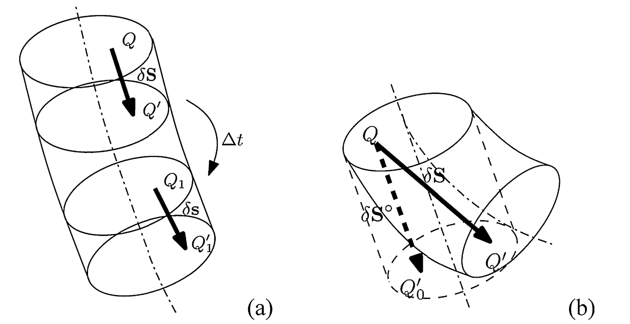

Fig. 1. The sketch of deformation analysis: (a) for the strain rate tensor, (b) for the strain tensor.

where $\boldsymbol{\kappa}(s) = \kappa_i\mathbf{d}_i$ is the curvature of the filament. We expand the element in the local coordinate as $\delta\mathbf{S} = \delta\tilde{l}_i\mathbf{d}_i$, and finally obtain the relation,

$$\begin{cases} \delta l_3 = (1 - \kappa_1 x_2 + \kappa_2 x_1)\delta\tilde{l}_3 \\ \delta l_2 = \delta\tilde{l}_2 - \kappa_3 x_1 \delta\tilde{l}_3 \\ \delta l_1 = \delta\tilde{l}_1 + \kappa_3 x_2 \delta\tilde{l}_3 \end{cases} \quad (2)$$

After a short time Δt, $\delta\mathbf{S}$ transformed into a new linear element $\delta\mathbf{s}$. A relation between two elements exists as $\delta\mathbf{s} = \delta\mathbf{S} + \Delta\mathbf{v}\Delta t$, where $\Delta\mathbf{v} = \mathbf{v}_{Q'} - \mathbf{v}_Q$ are the velocity difference between Q and Q'. The velocity of a material point is expressed by $\mathbf{v} = \nu\mathbf{d}_3 + \omega \times \rho + \mathbf{u}$. Here ν presents the magnitude of axis velocity, $\boldsymbol{\omega} = \omega_i\mathbf{d}_i$ is one half the vorticity at the jet axis for steady coiling and $\mathbf{u} = u_i\mathbf{d}_i$ is the lateral velocity in the cross section induced by incompressible condition. By the kinematic analysis of Ribe et al. [3,6], there are the relations $\omega_1 = \nu\kappa_1$, $\omega_2 = \nu\kappa_2$ and $\omega_3 = \nu\kappa_3 + \Omega$ holding. The definition of the rate of element length square requires,

$$d_t(|\delta\mathbf{s}|^2) = \lim_{\Delta t \to 0}[(|\delta\mathbf{s}|^2 - |\delta\mathbf{S}|^2)/\Delta t] = 2\Delta v_i \delta\tilde{l}_i \quad (3)$$

Here $\Delta\mathbf{v}$ can be calculated as,

$$\begin{aligned} \Delta\mathbf{v} &= \nu\kappa \times \mathbf{d}_3\delta l_3 + (\omega + \delta l_3\omega_i'\mathbf{d}_i + \delta l_3\kappa \times \omega) \times (\rho + \delta l_\alpha\mathbf{d}_\alpha + \delta l_3\kappa \times \rho) - \omega \times \rho + \delta l_3\kappa \times (u_\alpha\mathbf{d}_\alpha) + \delta l_\beta\partial_\beta u_\alpha\mathbf{d}_\alpha \\ &= [\nu\kappa \times \mathbf{d}_3 + \omega_i'\mathbf{d}_i \times \rho + \kappa \times (\omega \times \rho) + \kappa \times (u_\alpha\mathbf{d}_\alpha)]\delta l_3 + \omega \times (\delta l_\alpha\mathbf{d}_\alpha) + \delta l_\beta\partial_\beta u_\alpha\mathbf{d}_\alpha \end{aligned} \quad (4)$$

Unless otherwise noted, the primes appearing in (4) and other following expressions denote the differentiation with respect to s. By substituting (2) into (4) and omitting the higher order small terms, we can obtain the components of $\Delta\mathbf{v}$ as,

$$\begin{cases} \Delta v_1 = \partial_1 u_1\delta\tilde{l}_1 + (\partial_2 u_1 - \omega_3)\delta\tilde{l}_2 + (\nu\kappa_2 - \omega_3' x_2)\delta\tilde{l}_3 \\ \Delta v_2 = (\partial_1 u_2 + \omega_3)\delta\tilde{l}_1 + \partial_2 u_2\delta\tilde{l}_2 - (\nu\kappa_1 - \omega_3' x_1)\delta\tilde{l}_3 \\ \Delta v_3 = -\omega_2\delta\tilde{l}_1 + \omega_1\delta\tilde{l}_2 + (-B_2 x_1 + B_1 x_2)\delta\tilde{l}_3 \end{cases} \quad (5)$$

where $B_1 = \omega_1' - \omega_2\kappa_3 + \omega_3\kappa_2$ and $B_2 = \omega_2' + \omega_1\kappa_3 - \omega_3\kappa_1$. According to the theory of deformation analysis, the relation between the rate of square of line element length and the strain rate tensor is expressed by,

$$\begin{aligned} d_t(|\delta\mathbf{s}|^2) &= \delta\mathbf{S} \cdot 2\dot{\mathbf{E}} \cdot \delta\mathbf{S} \\ &= 2(\dot{E}_{11}\delta\tilde{l}_1^2 + \dot{E}_{22}\delta\tilde{l}_2^2 + \dot{E}_{33}\delta\tilde{l}_3^2 + \dot{\gamma}_{12}\delta\tilde{l}_1\delta\tilde{l}_2 + \dot{\gamma}_{13}\delta\tilde{l}_1\delta\tilde{l}_3 + \dot{\gamma}_{23}\delta\tilde{l}_2\delta\tilde{l}_3) \end{aligned} \quad (6)$$

Comparing (6) and (3), we obtain the normal strain rate components as $\dot{E}_{11} = \partial_1 u_1$, $\dot{E}_{22} = \partial_2 u_2$ and $\dot{E}_{33} = -B_2 x_1 + B_1 x_2$. The shear strain rate components are $\dot{\gamma}_{12} = 2\dot{E}_{12} = \partial_2 u_1 + \partial_1 u_2$, $\dot{\gamma}_{13} = 2\dot{E}_{13} = -\omega_3' x_2$ and $\dot{\gamma}_{23} = 2\dot{E}_{23} = \omega_3' x_1$. In order to solve out u_1 and u_2, two equations are needed, one of which is from that the first invariant of strain rate must be zero for the incompressible material and the

2.2. Constitutive relation and deformation analysis

2.2.1. Kelvin model

A Kelvin model is given by [10] $\boldsymbol{T} = -p\boldsymbol{I} + \boldsymbol{\sigma} + 2\mu\dot{\boldsymbol{E}} + 2G\boldsymbol{E}$, where $\mathbf{T}$ is the stress tensor of the filament. $-p\mathbf{I}$ exists due to the incompressible material condition and $\boldsymbol{\sigma}$ due to the neglect of shear and stretch between cross sections as a series of planes, the components σ_{11}, σ_{12} and σ_{22} of which are zeros. $\dot{\mathbf{E}}$ and $\mathbf{E}$ are the strain rate tensor and the strain tensor (both are equal to their deviators due to incompressible material), respectively, multiplied by viscosity μ and shear modulus G. Then the retardation time measuring viscoelasticity is defined by $\lambda = \mu/G$. A Kelvin model represents viscoelastic solid material. Below the strain rate tensor and the strain tensor will be determined by deformation analysis.

2.2.2. Strain rate tensor

The general idea we follow is that of Love [11]. As shown in Fig. 1(a), an arbitrary point Q within the filament is given by a position vector as, $\mathbf{r}_Q(s, x_1, x_2) = \boldsymbol{\gamma}_Q + \boldsymbol{\rho}$, where $\boldsymbol{\gamma}_Q(s)$ presents the position vector of the central line and $\boldsymbol{\rho}(s, x_1, x_2) = x_\alpha\mathbf{d}_\alpha(s)$ is a vector in plane from central line to Q. Another point Q' near Q is given by $\mathbf{r}_{Q'} = \gamma_Q(s + \delta l_3) + (x_\alpha + \delta l_\alpha)\mathbf{d}_\alpha(s + \delta l_3)$. Then a linear element can be expressed as $\delta\mathbf{S} = \mathbf{r}_{Q'} - \mathbf{r}_Q = \delta l_3(\mathbf{d}_3 + \boldsymbol{\kappa} \times \boldsymbol{\rho}) + \delta l_\alpha\mathbf{d}_\alpha$,