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# Fractional viscoplasticity

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ABSTRACT

In this paper we generalize the Perzyna's type viscoplasticity using fractional calculus. We call such model *fractional viscoplasticity*. The main objective of this research is to propose a new way of description of permanent deformation in a material body, especially under extreme dynamic conditions. In this approach the fractional calculus can be understood as a tool enabling the introduction of material heterogeneity/multi-scale effects to the constitutive model.

This newly developed phenomenological model is represented in the Euclidean space living more general setup for future work. The definition of the directions of a viscoplastic strains stated as a *fractional gradient* of plastic potential plays the fundamental role in the formulation. Moreover, the fractional gradient provides the non-associative plastic flow without necessity of additional potential assumption.



## 1. Introduction

Despite the fact that in most recent years micromechanical models describing material behavior are widely considered (cf. Stupkiewicz and Petryk, 2010; Coenen et al., 2012; Tasan et al., 2012) there is still need for new concepts in phenomenology particularly for models dedicated for extreme dynamic events (Sumelka and Łodygowski, 2011; Rusinek et al., 2007). In principal, micromechanics provides naturally deeper insight into the physical phenomena being considered but from the other point of view it appears that such formulations are still not suitable for extreme dynamic processes where wave effects play fundamental role and also current software/hardware capabilities are not good enough. On the other hand the most important drawback in phenomenology, in contrary to micromechanics, is that many material parameters need to be identified for practical applications when many phenomena are considered (e.g. thermo-mechanical coupling including anisotropic description of damage or phase transformations) (Eftis et al., 2003; Glema et al., 2009; Sumelka, 2009). Hence, the crucial tasks for research in the are of phenomenology is to simplify material functions considered or develop additional techniques such as soft computing reducing the number of considered material parameters (Sumelka and Łodygowski, 2013a). It is shown that to some extend fractional viscoplasticity can be viewed as a solution to described circumstances. In other words fractional calculus can be understood as a tool for introduction of material heterogeneity/multi-scale effects to the constitutive model (Sumelka, 2013a).

Independently of chosen technique describing experimentally observed body behavior the fundamental question arises: do we use correct mathematical tools for the description of the material body deformation? More precisely in aspect of the subject of this paper: are commonly used differential operators in the particular model correctly assumed to be of integer order or one should choose more general one, namely the differential operators of an arbitrary order? The answer to such question is not obvious. Considering many successful applications of fractional calculus in Fluid Flow, Rheology, Dynamical Processes in Self-Similar and Porous Structures, Diffusive Transport Akin to Diffusion, Electrical Networks, Probability and Statistics, Control Theory of Dynamical Systems, Viscoelasticity, Electrochemistry of Corrosion, Chemical Physics, Optics and others (Podlubny, 1999; Tarasov, 2008; Mainardi, 2010 and cited therein) one can be more than sure that in the theories describing permanent deformation of a body, such as viscoplasticity/plasticity, the use of fractional calculus should be appropriate.

In several papers (Sumelka, 2012a,b) the original idea of *fractional viscoplasticity* is introduced. Fractional viscoplasticity is generalization of classical Perzyna's type viscoplasticity (Perzyna, 1963) using fractional calculus. The fundamental role in the formulation plays the definition of the directions of a viscoplastic strains given as a *fractional gradient* of plastic potential. In this way one obtains a flexible tool that controls viscoplastic strain evolution (magnitude and directions) without necessity of adding explicitly new phenomena to the constitutive structure. Moreover, by introducing the new parameter to the model (order of derivative) we simultaneously obtain the non-associative plastic flow

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(in general) without necessity of additional potential assumption. This way we we decrease the number of material functions and simultaneous parameters. Classical Perzyna's solution is obtained as a special case - when order of fractional gradient is assumed to be equal to one.

As opposed to the previous works (Sumelka, 2012a,b) this paper provides complete description of this topic. Different fractional differential operator for fractional gradient of plastic potential is applied as well. Thus, fractional viscoplasticity can be redefined using different definitions of fractional derivative. In this sense there should exist optimal definition of derivative for a specific material, such as steel, rubber or concrete. As an example, the definition of classical Riesz–Feller fractional operator (Feller, 1952) (not discussed here) has an origin in processes with Lévy stable probability distribution. In this sense it should be possible to define fractional differential operator in such a way that it carries information about the distribution of grains sizes in a particular metal.

The paper is divided into three main parts.

In Section 2 fundamental concepts of fractional calculus are presented to justify assumptions imposed during the fractional viscoplasticity definition.

The fractional viscoplasticity in Euclidean space, leaving more general setup for future work, is defined in Section 3 along with the fractional viscoplatic strain gradient of Caputo's type. Because the fractional derivative is defined on interval (contrary to standard definition of derivative in a point) so called "short memory" principle (Podlubny, 1999) is utilized to make bounds of this interval with clear physical interpretation.

In Section 4 illustrative example showing the dependence of the direction of the viscoplastic flow plotted against the order of fractional gradient is discussed to prove that in general the non-associative plastic flow without necessity of additional potential assumption is obtained.

## 2. Fractional calculus – fundamental concepts

The theory of derivatives of non-integer order was initiated on 30th of September 1695 when Leibniz showed his concerns about the L'Hospital's derivative of order one and a half (Leibniz, 1962). The breakthrough sentence by Leibniz stated: "It will lead to a paradox from which one day useful consequences will be drawn". Since that day fractional calculus became an individual branch of pure mathematics with many successful applications. It was discussed in many comprehensive encyclopedic-type monographs e.g. (Samko et al., 1993; Podlubny, 1999; Kilbas et al., 2006; Leszczyński, 2011).

Although there are numerous definitions for fractional differential operators they share the common attribute: they are defined on an interval in contrary to the integer order differential operators defined in a single point. The most commonly used are those defined by generalization of $n$-fold integration or $n$-fold derivative. To understand the idea let us consider the $n$-fold integration of a function $f$ which is given by

$$f^{(-n)}(t) = \frac{1}{\Gamma(n)} \int_a^t (t-\tau)^{n-1} f(\tau) d\tau, \quad t > a, \quad n \in \mathbb{N}, \tag{1}$$

where $\Gamma$ is the Euler gamma function defined as ($a$ is arbitrary)

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt. \tag{2}$$

Notice that if in Eq. (2) we apply $\alpha = n \in \mathbb{N}\backslash\{0\}$ we have $\Gamma(n) = (n-1)!$. Now, if we replace in Eq. (1) $n$ with an arbitrary $\alpha > 0$ we obtain (left) fractional integral operator in Riemann–Liouville (RL) sense

$$_a I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad t > a, \quad n \in \mathbb{R}^+. \tag{3}$$

Based on relation Eq. (3) one can define the following fractional derivatives (left sided)

$$^{RL}_{a}D_t^\alpha f(t) = D^m ({}_a I_t^{m-\alpha} f)(t), \tag{4}$$

$$^{C}_{a}D_t^\alpha f(t) = {}_a I_t^{m-\alpha} (D^m f)(t), \tag{5}$$

where $m = [\alpha] + 1$, $^{RL}_{a}D_t^\alpha f(t)$ and $^{C}_{a}D_t^\alpha f(t)$ defines fractional derivatives in Riemann–Liouville (RL) and Caputo (C) sense, respectively.

As already mentioned the derivatives of an arbitrary order (even complex) are defined on an interval, thus one can define so called left and right sided derivatives. Considering Caputo (C) type derivative (the one used during fractional viscoplasticity definition) as an example, the explicit definitions are: left-sided Caputo's derivative for $t > a$ and $n = [\alpha] + 1$

$$^{C}_{a}D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau; \tag{6}$$

and right-sided Caputo derivative for $t < b$ and $n = [\alpha] + 1$

$$^{C}_{t}D_b^\alpha f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b \frac{f^{(n)}(\tau)}{(\tau-t)^{\alpha-n+1}} d\tau, \tag{7}$$

where $\alpha > 0$ denotes the real order of the derivative, $D$ denotes 'derivative' and $a, t, b$ are so called terminals. Notice that both definitions include integration over the interval $(a, t)$ or $(t, b)$, respectively. The terminals $a$ and $b$ can be chosen arbitrarily. Nevertheless, for fractional viscoplasticity definition we will use so called "short memory" principle (Podlubny, 1999) for terminal definition for clearer physical interpretation. It is clear that terminals must not be constant during the deformation – e.g. they can be a function of state variables.

As an illustrative example let us consider Caputo derivative of a power function $f(t) = (t-a)^\nu$, in this case we have

$$^{C}_{a}D_t^\alpha (t-a)^\nu = \frac{\Gamma(\nu+1)}{\Gamma(-\alpha+\nu+1)} (t-a)^{\nu-\alpha}, \tag{8}$$

or optionally for $f(t) = (b-t)^\nu$ we have

$$^{C}_{t}D_b^\alpha (b-t)^\nu = \frac{\Gamma(\nu+1)}{\Gamma(-\alpha+\nu+1)} (b-t)^{\nu-\alpha}, \tag{9}$$

and important result in case $f(t) = C = const$.

$$^{C}_{t}D_b^\alpha C = {}^{C}_{a}D_t^\alpha C = 0. \tag{10}$$

In general any type of the fractional derivative of a constant function is **not** equal to zero (Caputo's derivative is an exception Eq. (10)). It is also fundamental that using Caputo's type derivative one needs standard (like in the classical differential equations) initial and/or boundary conditions, while for other types of fractional derivatives (e.g. RL) they are of a different type dependently of chosen definition.

Finally, let us define the Caputo's type derivative for interval $t \in (a, b)$. We call such derivative Riesz–Caputo (RC) derivative cf. (Frederico and Torres, 2010). This type of fractional derivative is crucial for further definition of directions of viscoplastic strains. Since any linear combination of derivatives Eqs. (6) and (7) defines new derivative (Samko et al., 1993), we put for $t \in (a, b) \subseteq \mathbb{R}$, $\alpha > 0$ and $n - 1 < \alpha < n$ (Agrawal, 2007) (when $\alpha$ is an integer, the usual definition of a derivative is used cf. (Agrawal, 2007; Frederico and Torres, 2010))

$$^{RC}_{a}D_b^\alpha f(t) = \frac{1}{2} \left( {}^{C}_{a}D_t^\alpha f(t) + (-1)^n {}^{C}_{t}D_b^\alpha f(t) \right). \tag{11}$$