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Point force and dipole solutions in the inhomogeneous half-plane under time-harmonic conditions

Tsviatko V. Rangelov[a,*], George D. Manolis[b]

[a] Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Sofia 1113, Bulgaria
[b] Department of Civil Engineering, Aristotle University, Thessaloniki GR-54124, Greece

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ABSTRACT

In this paper, we present fundamental solutions for the inhomogeneous half-plane under anti-plane strain conditions subjected to a point force and two dipoles. Time-harmonic conditions are assumed to hold, while the boundary conditions comprise a traction-free horizontal surface plus the Sommerfeld radiation condition. The aforementioned fundamental solutions are derived for two special types of continuous material inhomogeneity, whereby the shear modulus and the density vary either as an exponential function or as a quadratic polynomial with respect to depth. These solutions converge to their static equivalents as the frequency of vibration approaches zero, and collapse to the ones corresponding to the homogeneous half-plane when the inhomogeneity parameter is set to zero. Finally, a numerical example serves to illustrate the fundamental solutions obtained herein.

1. Introduction

The recovery of fundamental solutions (or Green's functions) for some canonical problems in elastodynamics (Kausel, 2006) has spurred the development of boundary integral equation methods as one of the most successful numerical tools for solving wave propagation and scattering problems in infinite, semi-infinite and finite domains (Dominguez, 1993). Fundamental solutions are most often particular solutions of the differential operator in question for point forces in space and time applied to unbounded domains. In turn, they serve as kernel functions in integral equation formulations that are redefined for both source (where the point load is applied) and receiver (where the signal is measured) ascending to the surface of the elastic body in question. These singular boundary integral equations (BIE) are defined in the Cauchy principal value sense and subsequent numerical discretization of all surfaces in question allows for the solution of boundary value problems with important applications in mechanics, geotechnics, seismology, etc. Green's functions can of course be used directly in numerical applications, giving rise to specialized methods such as the wave number integration method. Advantages of boundary element methods, namely the numerical version of BIE, over domain-type methods such as finite elements stem from the fact that surface-only meshes are necessary, the radiation condition is automatically accounted for and use of the Green's functions yields high accuracy in the results obtained.

Green's function for elastic and isotropic continua is a classical problem, see Kausel (2012). The first important step in their derivation was the introduction of a free surface so as to model the half-space (i.e., Lamb's problem, early 20th century), opening the door for important applications involving the surface of the earth and the free surface of seas. The next step was to introduce layering and other types of inhomogeneities in the elastic continuum so as to realistically model natural or certain categories of man-made materials, see Ewing et al. (1957). Since then, many specialized fundamental solutions have been produced by considering variable mechanical properties, anisotropy, coupled fields such as poroelasticity and so on (see Rangelov et al., 2005 for a more detailed account).

The present work is a continuation of an earlier derivation (Rangelov and Manolis, 2010) of Green's functions for the scalar wave equation defined in the half-space with a quadratic-type of

* Corresponding author. Tel.: +359 02979 2845; fax: +359 02971 3649.
E-mail addresses: rangelov@math.bas.bg, rangelov@yahoo.com (T.V. Rangelov).

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material inhomogeneity, assumed to hold proportionally for both shear modulus and density. In here, we derive the contribution of dipoles to this Green's function that was originally defined for a point source, and furthermore examine a material inhomogeneity of the exponential-type.

The scalar wave equation for non-homogeneous, elastic continuous media has been studied by many researchers. In here, we mention the early work of Bhattacharya (1970) who presented a rather thorough classification of inhomogeneities in either the material velocity, the elastic modulus or the density, for which analytical solutions are possible. This type of approach, whereby one investigates possible mathematical variations of the material parameters that lead to standard types of partial differential equations, and hence to known solutions in terms of special functions, has since been followed by other researchers. For instance, Kuvshinov and Mulder (2006) derived an exact solution for a linear velocity profile and a density obeying a power law.

Green's functions for the inhomogenous, elastic half-plane need to satisfy an extra boundary condition in the form of a traction-free horizontal surface. The usual approach is to consider this constraint directly, which adds extra correction terms to the Green's function for the full-space. These terms tend to decay rapidly for points far from the free surface, causing the solution to converge to that for the full space. In this context, we mention the work of Vrettos (1985) who derived a Green's function for the elastic half-space corresponding to a bounded exponential variation of the shear modulus with respect to the depth coordinate. It is also possible to construct approximate Green's functions by superposition of more elementary forms, such as the Gaussian beam which is a ray-type approximation. For instance, Wu (1985) approximated the Green's function for a smoothly inhomogeneous medium by a bundle of overlapped Gaussian beams, a representation that is similar to a uniform asymptotic representation.

When geophysical applications are of interest, it becomes necessary to introduce more realistic material representations, and this primarily is anisotropy. For instance, Psencik (1998) derived a Green's function comprising a zero order plus a first order term for the unbounded inhomogeneous, and weakly anisotropic, continuum. These terms are derived from an asymptotic solution of the governing equations expanded in terms of two small parameters, one used in conjunction with the standard ray method and a second used as a measure of the deviation between the anisotropic material and the background isotropic one. A more general solution for the Green's tensor corresponding to a fully anisotropic and inhomogeneous medium, but as a far-field frequency approximation, was derived by Ben-Menahem and Sena (1991).

Finally, as one looks at BEM formulations in order to exploit available Green's functions, we mention the related work of Brun et al. (2003a,b) on non-linear elasticity under quasi-static conditions. These material models allow for the analysis of instability phenomena in the material, such as the computation of bifurcation loads and their deformation modes. These derivations were extended by Bigoni et al. (2007) to cover dynamic problems in the form of time-harmonic, small amplitude vibrations of pre-stressed elastic solids that are incompressible and orthotropic. This is basically the motivation for deriving specialized Green's functions, because a BEM formulation will allow for the efficient solution of complex BVP of engineering importance.

In this work, we present a Green's function for the half-plane under anti-plane strain conditions for two specific types of continuous inhomogeneity, namely quadratic and exponential dependence of the material properties on the depth coordinate. The methodology follows the procedure initiated in Manolis and Shaw (1996) for the full plane, whereby an algebraic transformation of the equations of motion produces a series of extra terms in the differential operator. Subsequent annulment of these terms yields a system of constraint equations on admissible forms of inhomogeneity, which render the remaining problem mathematically similar to retrieving fundamental solutions for a homogeneous continuum. In what follows, we derived solutions for a point source and dipoles (in the sense of R. Mindlin) in this special type of continuously inhomogeneous half-plane under time harmonic conditions. Limiting forms of these Green's functions for quasi-static conditions and for the homogeneous continuum are also produced. Finally, a numerical example for a soil material serves to illustrate the use of these solutions and to juxtapose the difference between wave propagation in continuously homogeneous versus inhomogeneous media.

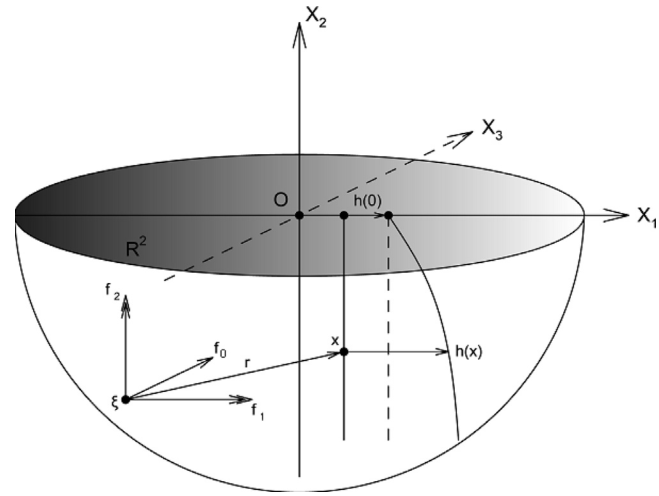

Fig. 1. Harmonic wave propagation due to a point force and dipoles in the inhomogeneous half-plane under anti-plane strain conditions.

2. Statement of the problem and solution method

We introduce a Cartesian coordinate system Ox_1x_2 and define the half-plane $R^2_- = \{x = (x_1, x_2), x_2 < 0\}$. Function $h(x)$ is also defined in R^2_-, depends only on x_2 and $h(x) > 0$, $h(x) \in C^2(R^2_-)$, as shown in Fig. 1. With $\mu_0 > 0$, $\rho_0 > 0$ as the reference values for the shear modulus and the density, respectively, we further define $\mu(x) = h(x)\mu_0$, $\rho(x) = h(x)\rho_0$.

The constitutive equations for the anti-plane case are

$$\sigma_i(x, \omega) = \mu(x)u_{,i}(x, \omega). \quad (1)$$

Here, $\sigma_i(x, \omega)$ are the out-of-plane shear stress components, $\mu(x)$ is now the position dependent shear modulus, $u(x, \omega)$ is the out-of-plane displacement, index $i = 1, 2$, ω is the frequency of vibration and commas denote spatial derivatives. The corresponding equation of motion in the frequency domain is

$$L^h(u) \equiv \sigma_{i,i}(x, \omega) + \rho(x)\omega^2 u(x, \omega) = f(x) \quad x \in R^2_-, \quad (2)$$

where $\rho(x)$ is the position dependent density, $f(x)$ is the external body force and summation under repeated indexes is implied.

Assume that f is a force along the Ox_3-direction with support at a point $\xi \in R^2_-$ in the form

$$f(x, \xi) = f_0\delta(x, \xi) + f_j\delta_{,j}(x, \xi) \quad x, \xi \in R^2_-, \quad (3)$$

where δ is Dirac's function and f_0 is a force per unit volume, while f_1 and f_2 are moments per unit volume.

Consider next a traction free boundary condition along the free surface $x_2 = 0$

$$t(x, \omega)|_{x_2=0} = 0 \quad (4)$$

where $t = \sigma_i n_i$, is the out-of-plane tangential traction, and $n = (n_1, n_2)$ is the unit normal vector.