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Stress gradient continuum theory

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ABSTRACT

A stress gradient continuum theory is presented that fundamentally differs from the well-established strain gradient model. It is based on the assumption that the deviatoric part of the gradient of the Cauchy stress tensor can contribute to the free energy density of solid materials. It requires the introduction of so-called micro-displacement degrees of freedom in addition to the usual displacement components. An isotropic linear elasticity theory is worked out for two-dimensional stress gradient media. The analytical solution of a simple boundary value problem illustrates the essential differences between stress and strain gradient models.

1. Introduction

Much attention has been dedicated to strain gradient effects in continuum mechanics and materials sciences in the last fifty years, since the pioneering work of Toupin (1962) and Mindlin (1965). The second gradient theory represents an extension of the classical Cauchy continuum by incorporating the effect of the second gradient of the displacement field into the balance and constitutive equations of the medium, in addition to the usual first gradient of the displacement. It must be noted that the second gradient of the displacement theory and the strain gradient model represent the same continuum, due to compatibility conditions, as shown by Mindlin and Eshel (1968). Higher order stresses, called hyperstresses or double stresses, must be included in the theory as the quantities conjugate to the components of the second gradient of displacement. This results in an extended balance of momentum equation and additional boundary conditions. These equations have been derived first by Toupin and Mindlin using variations of the elastic energy, and then by Germain (1973a) by means of the method of virtual power. A derivation *à la Cauchy*, i.e. based on the representation of generalized contact forces, was established more recently by Noll and Virga (1990) and Dell'Isola and Seppecher (1995, 1997), due to the fact that the Neumann conditions are rather intricate in a second gradient medium.

In contrast, the role of stress gradients has been the subject of little attention, if one excepts its introduction in fatigue crack initiation models at notches and holes of various sizes as studied in the engineering community (Bascoul and Maso, 1981; Lahellec et al., 2005). More recently, a stress-gradient based criterion has been proposed for dislocation nucleation in crystals at a nano-scale (Acharya and Miller, 2004).

Regarding continuum mechanics, there is a long-standing misconception or, at least, ambiguity going through the whole literature on generalized continua, that implicitly considers that the strain gradient theory can also be regarded as a stress gradient model. The stress gradient can be found in Aifantis gradient elasticity model (Aifantis, 1992, 2009; Ru and Aifantis, 1993; Lazar et al., 2006) in the form:

$$\underset{\sim}{\boldsymbol{\tau}} = \underset{\sim}{\boldsymbol{\sigma}} - c\nabla^2 \underset{\sim}{\boldsymbol{\sigma}} \quad (1)$$

where $\underset{\sim}{\boldsymbol{\tau}}$ is an effective stress tensor whose divergence vanishes in the absence of body forces and c is a material parameter associated with a characteristic length. In a Cartesian orthonormal coordinate system the Laplace operator is applied to each component of the matrix. The Laplace term arises as the divergence of the gradient of the stress field. However, it can be shown that the presence of the stress gradient in this model is the result of a specific constitutive assumption made in Mindlin's strain gradient elasticity (Forest and Aifantis, 2010). Accordingly, Aifantis gradient elasticity must be considered as a strain gradient model.

As a result, the question arises whether it is possible to formulate a stress gradient continuum theory describing

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size-dependent material properties and how much it may differ from the well-established strain gradient model. Generalized continuum theories include strain gradient, micromorphic and non-local models (Eringen, 1999, 2002) that introduce higher order strain gradients, additional degrees of freedom or non-local kernels, but not explicitly the stress gradient as a primary variable. To the knowledge of the authors, a stress gradient continuum theory does not exist in the literature. The objective of the present work is to establish the framework of such a stress gradient theory and to illustrate the predicted behavior in the case of linear isotropic elasticity. In particular we will prove that this theory fundamentally differs from Mindlin's strain gradient model: stress gradient and strain gradient models are two distinct representations of the continuum.

The presented stress gradient theory for the 3D continuum is similar to the so-called bending-gradient theory recently proposed by Lebée and Sab (2011a,b) for out-of-plane loaded elastic thick laminated plates. In this plate theory, the stress energy density is a function of the local bending moment and its gradient. Moreover, these authors show that the well-known Reissner plate theory (Reissner, 1945) for out-of-plane loaded elastic thick homogeneous plates actually is a degenerated case of their bending-gradient theory. In the bending-gradient theory the stress energy density is a function of the local bending moment and of the spherical part of its gradient which coincides with the classical shear forces, see also (Cecchi and Sab, 2007; Nguyen et al., 2007, 2008).

A systematic comparison of the new model will be drawn with Mindlin's second gradient theory and Germain's general micromorphic theory (Germain, 1973b). The pros and the cons of each model will be addressed at different stages of the discussion. In particular, both computational and physical, or more precisely micro-mechanical, arguments will be raised to characterize the new approach.

For the sake of brevity, the theory is developed within the small deformation framework and under static conditions. A first construction of the theory is proposed in Section 2 for elastic stress gradient solids. The general theory, independent of the constitutive behavior, is presented based on the method of virtual power in Section 3. A two-dimensional linear isotropic elasticity theory is formulated in Section 4. Finally, the responses of the stress gradient and strain gradient continua are compared in Section 5 in the case of a generic boundary value problem involving periodic body forces.

Tensors of zeroth, first, second, third and fourth ranks are respectively denoted by a, $\underline{\boldsymbol{a}}$, $\underset{\sim}{\boldsymbol{a}}$, $\underset{\approx}{\underline{\boldsymbol{a}}}$ (or $\underset{\sim}{\underline{\boldsymbol{a}}}$) and $\underset{\approx}{\boldsymbol{a}}$. The intrinsic notation is usually complemented by the index notation to avoid any confusion. The tensor product is denoted by $\otimes$. We also define the symmetrized tensor product using the following notations:

$$\underline{\boldsymbol{a}} \overset{s}{\otimes} \underline{\boldsymbol{b}} = \frac{1}{2}(\underline{\boldsymbol{a}} \otimes \underline{\boldsymbol{b}} + \underline{\boldsymbol{b}} \otimes \underline{\boldsymbol{a}}), \quad a_{(i}b_{j)} = \frac{1}{2}(a_i b_j + a_j b_i) \tag{2}$$

The nabla operator is denoted by ∇ and operates as follows on a vector field, in a Cartesian orthonormal basis $(\underline{\boldsymbol{e}}_1, \underline{\boldsymbol{e}}_2, \underline{\boldsymbol{e}}_3)$:

$$\underline{\boldsymbol{u}}(\underline{\boldsymbol{x}}) \otimes \nabla = \frac{\partial u_i}{\partial x_j} \underline{\boldsymbol{e}}_i \otimes \underline{\boldsymbol{e}}_j = u_{i,j} \underline{\boldsymbol{e}}_i \otimes \underline{\boldsymbol{e}}_j \tag{3}$$

The Cauchy stress tensor is denoted by $\underset{\sim}{\boldsymbol{\sigma}}$ and has the following components:

$$\underset{\sim}{\boldsymbol{\sigma}} = \sigma_{ij} \underline{\boldsymbol{e}}_i \otimes \underline{\boldsymbol{e}}_j \tag{4}$$

The stress gradient tensor is defined as

$$\underset{\sim}{\boldsymbol{\sigma}} \otimes \nabla = \sigma_{ij,k} \underline{\boldsymbol{e}}_i \otimes \underline{\boldsymbol{e}}_j \otimes \underline{\boldsymbol{e}}_k \tag{5}$$

Its divergence is the vector

$$\underset{\sim}{\boldsymbol{\sigma}} \cdot \nabla = \sigma_{ij,j} \underline{\boldsymbol{e}}_i \tag{6}$$

2. Formulation of a stress gradient elasticity model

2.1. Algebra of deviatoric third rank tensors

The stress gradient tensor is the third rank tensor defined by Eq. (5). Its components are symmetric with respect to the two first indices. In this work, the space of third rank tensors that are symmetric with respect to the first two indices is denoted by $\mathcal{R}$. It is a vector space of dimension 18 which is endowed with the scalar product:

$$\underset{\approx}{\underline{\boldsymbol{R}}} \therefore \underset{\approx}{\underline{\boldsymbol{R}}} = R_{ijk} R_{ijk}, \quad \forall \underset{\approx}{\underline{\boldsymbol{R}}} \in \mathcal{R} \tag{7}$$

Each tensor $\underset{\approx}{\underline{\boldsymbol{R}}} \in \mathcal{R}$ can then be decomposed into a spherical part $\underset{\approx}{\underline{\boldsymbol{R}}}^s \in \mathcal{S} \subset \mathcal{R}$ and a deviatoric part $\underset{\approx}{\underline{\boldsymbol{R}}}^d \in \mathcal{D} \subset \mathcal{R}$:

$$\underset{\approx}{\underline{\boldsymbol{R}}} = \underset{\approx}{\underline{\boldsymbol{R}}}^s + \underset{\approx}{\underline{\boldsymbol{R}}}^d \tag{8}$$

with

$$R^s_{ijk} = \frac{1}{4}(R_{ilm}\delta_{lm}\delta_{jk} + R_{jlm}\delta_{lm}\delta_{ik}) \tag{9}$$

Here, the space $\mathcal{D}$ is the subset of $\mathcal{R}$ containing the deviatoric elements $\underset{\approx}{\underline{\boldsymbol{R}}}$ such that

$$\underset{\approx}{\underline{\boldsymbol{R}}} : \underset{\sim}{\boldsymbol{1}} = 0, \quad R_{ijk}\delta_{jk} = 0 \tag{10}$$

where $\underset{\sim}{\boldsymbol{1}}$ is the second rank identity tensor and δ_{ij} is the Kronecker symbol. It follows that $\mathcal{S} = \mathcal{D}^{\perp}$ and $\mathcal{R} = \mathcal{D} \oplus \mathcal{S}$.

We finally note that the spherical part of the stress gradient is directly related to the divergence of the stress tensor by

$$(\underset{\sim}{\boldsymbol{\sigma}} \otimes \nabla)^s_{ijk} = \frac{1}{4}(\sigma_{im,m}\delta_{jk} + \sigma_{jm,m}\delta_{ik}) \tag{11}$$

or equivalently,

$$(\underset{\sim}{\boldsymbol{\sigma}} \otimes \nabla) : \underset{\sim}{\boldsymbol{1}} = (\underset{\sim}{\boldsymbol{\sigma}} \otimes \nabla)^s : \underset{\sim}{\boldsymbol{1}} = \underset{\sim}{\boldsymbol{\sigma}} \cdot \nabla \tag{12}$$

The previous definitions are valid in the physical three-dimensional space. However, we will also need expressions in the two-dimensional space. In the purely two-dimensional case, the formula (9) must be replaced by

$$R^s_{ijk} = \frac{1}{3}(R_{ilm}\delta_{lm}\delta_{jk} + R_{jlm}\delta_{lm}\delta_{ik})$$

where the indices i, j, k only take the values 1, 2. In the two-dimensional case, the matrix form of the decomposition (8) becomes

$$\begin{bmatrix} R_{111} \\ R_{122} \\ R_{221} \\ R_{222} \\ R_{211} \\ R_{112} \end{bmatrix} = \begin{bmatrix} R^s_{111} \\ R^s_{122} \\ R^s_{221} \\ R^s_{222} \\ R^s_{211} \\ R^s_{112} \end{bmatrix} + \begin{bmatrix} R^d_{111} \\ R^d_{122} \\ R^d_{221} \\ R^d_{222} \\ R^d_{211} \\ R^d_{112} \end{bmatrix} \tag{13}$$