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On new symplectic superposition method for exact bending solutions of rectangular cantilever thin plates

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ABSTRACT

A novel superposition method based on the symplectic geometry approach is presented for exact bending analysis of rectangular cantilever thin plates. The basic equations for rectangular thin plate are first transferred into Hamilton canonical equations. By the symplectic geometry method, the analytic solutions to some problems for plates with slidingly supported edges are derived. Then the exact bending solutions of rectangular cantilever thin plates are obtained using the method of superposition. The symplectic superposition method developed in this paper is completely rational compared with the conventional analytical ones because the predetermination of deflection functions, which is indispensable in existing methods, is dispelled.

1. Introduction

The bending of rectangular thin plates with various combinations of boundary conditions has been investigated for many years by different authors. It is well known that explicit analytic solutions of rectangular thin plates are available only for those with two opposite sides simply supported (i.e. Navier's solution, Levy's solution, etc.) while it is, so far, difficult to get the solutions which exactly satisfy both the partial differential equation and other boundary conditions of a plate. Accordingly, various methods have been studied. One of the most commonly used methods for exact bending solutions of rectangular thin plates is the superposition method (Timoshenko and Woinowsky-Krieger, 1959; Huang and Conway, 1952; Chang, 1980, 1981, 1984). The technique of Fourier series expansion is another procedure for accurate bending analysis of plates (Khalili et al., 2005). Besides, a number of numerical methods have been frequently adopted in analyzing plate bending problems such as the finite difference method (Holl, 1937; Barton, 1948; MacNeal, 1951; Nash, 1952), finite element method (Zienkiewicz and Cheung, 1964), finite strip method (Cheung, 1976), method of discrete singular convolution (Civalek, 2007), method of differential quadrature (Civalek, 2004).

Cantilever thin plate is an important structural element while its bending has been one of the most difficult problems in the theory of elastic thin plate for the complexity in both the governing equation and the boundary conditions. Consequently, some approximate methods were utilized for the problem. The method of finite difference was firstly used to solve a cantilever plate with concentrated edge load by Holl (1937). The problem is also solved by Barton (1948), MacNeal (1951), Livesley and Birchall (1956) separately with the same method. Some other approximate analysis of the bending of a rectangular cantilever plate by uniform normal pressure was presented by Nash (1952). The generalized variational principle was applied to rectangular thin plates by Shu and Shih (1957), and the principle was then used by Plass et al. (1962) for deflection and vibration problems of cantilever plates. Leissa and Niedenfuhr (1962) obtained the solution for uniformly loaded cantilevered square plates using the technique of point matching and the Rayleigh–Ritz method. In addition, Chang (1980, 1981, 1984) derived series solutions for the bending of both uniformly loaded and concentrated loaded rectangular cantilever plates by using the method of superposition, which involved a skillful superposition of several problems, yet used smart trial functions.

In the present paper, a novel superposition method based on the symplectic geometry approach (Zhong and Williams, 1993; Yao and Zhong, 2002; Zhong and Li, 2009; Liu and Li, 2010) is developed to obtain exact bending solutions of rectangular cantilever thin plates under arbitrary loading. Unlike the traditional semi-inverse approaches in classical plate analysis employed by Timoshenko and Woinowsky-Krieger (1959) and others such as Chang (1980, 1981, 1984), where a trial deflection function has to be predetermined, the analysis here is completely rational without any trial functions. The procedure of solution presented enables one to acquire exact solutions for more problems of plates which have to hitherto be analyzed using the semi-inverse method or approximate approaches. It can be not only applied to other combinations of

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boundary conditions but also further extended to the problems of moderately thick plates as well as buckling, vibration, etc.

2. Hamilton dual equation for rectangular thin plates

The coordinate system of a rectangular cantilever thin plate under consideration is illustrated in Fig. 1a, where $0 \le x \le a$ and $0 \le y \le b$. The governing equations of the plate are

$$\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}-Q_x=0 \tag{1}$$

$$\frac{\partial M_y}{\partial y}+\frac{\partial M_{xy}}{\partial x}-Q_y=0 \tag{2}$$

$$\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+q=0 \tag{3}$$

$$M_x=-D\left(\frac{\partial^2 W}{\partial x^2}+\nu\frac{\partial^2 W}{\partial y^2}\right)$$
$$M_y=-D\left(\frac{\partial^2 W}{\partial y^2}+\nu\frac{\partial^2 W}{\partial x^2}\right) \tag{4a–c}$$
$$M_{xy}=-D(1-\nu)\frac{\partial^2 W}{\partial x\partial y}$$

The internal forces of the plate are represented as

$$Q_x=\frac{-D\partial(\nabla^2 W)}{\partial x};\quad Q_y=\frac{-D\partial(\nabla^2 W)}{\partial y} \tag{5a,b}$$

$$V_x=Q_x+\frac{\partial M_{xy}}{\partial y};\quad V_y=Q_y+\frac{\partial M_{xy}}{\partial x} \tag{6a,b}$$

where W is the transverse deflection of plate midplane, D is the flexural rigidity, q is the distributed transverse load, M_x, M_y, M_{xy}, Q_x, Q_y, V_x and V_y are the bending moments, torsional moment, shear forces and total shear forces, respectively.

From Eqs. (6a,b) and (3), we have

$$\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}-2\frac{\partial M_{xy}}{\partial x\partial y}+q=0 \tag{7}$$

Putting

$$\frac{\partial W}{\partial y}=\theta \tag{8}$$

Eq. (4b) yields

$$\frac{\partial\theta}{\partial y}=\frac{-\nu\partial^2 W}{\partial x^2}-\frac{M_y}{D} \tag{9}$$

Eq. (4c) is rewritten as

$$M_{xy}=\frac{-D(1-\nu)\partial\theta}{\partial x} \tag{10}$$

From Eqs. (4b,c), (5a), (6a) and (7), we find

$$\frac{\partial V_y}{\partial y}=D\left(1-\nu^2\right)\frac{\partial^4 W}{\partial x^4}-\nu\frac{\partial^2 M_y}{\partial x^2}-q \tag{11}$$

From Eqs. (6b), (2) and (10), we obtain

$$\frac{\partial M_y}{\partial y}=V_y+2D(1-\nu)\frac{\partial^2\theta}{\partial x^2} \tag{12}$$

Putting $V_y = -T$, Eqs. (8), (9), (11) and (12) are represented in the matrix form as

$$\dot{\mathbf{Z}}=\mathbf{HZ}+\mathbf{f} \tag{13}$$

where $\mathbf{H}=\begin{bmatrix}\mathbf{F} & -\mathbf{G}\\ -\mathbf{Q} & -\mathbf{F}^T\end{bmatrix}$, $\mathbf{F}=\begin{bmatrix}0 & 1\\ -\nu\partial^2/\partial x^2 & 0\end{bmatrix}$, $\mathbf{G}=\begin{bmatrix}0 & 0\\ 0 & 1/D\end{bmatrix}$, $\mathbf{Q}=D(1-\nu)\begin{bmatrix}(1+\nu)\partial^4/\partial x^4 & 0\\ 0 & -2\partial^2/\partial x^2\end{bmatrix}$, $\mathbf{Z}=[W,\theta,T,M_y]^T$, $\mathbf{f}=[0,0,q,0]^T$, the over dot denotes differentiation with respect to y. Observing $\mathbf{H}^T=\mathbf{JHJ}$, where $\mathbf{J}=\begin{bmatrix}0 & \mathbf{I}_2\\ -\mathbf{I}_2 & 0\end{bmatrix}$ is the symplectic matrix in which $\mathbf{I}_2$ is 2×2 unit matrix, thus $\mathbf{H}$ is a Hamiltonian operator matrix and Eq. (13) is the Hamiltonian dual equation for the plate.

3. Symplectic analytic solution of a plate with one edge slidingly supported and the opposite edge simply supported

The homogeneous equation of Eq. (13) is

$$\dot{\mathbf{Z}}=\mathbf{HZ} \tag{14}$$

According to the symplectic approach, applying the method of separation of variables to $\mathbf{Z}$ yields

$$\mathbf{Z}=\mathbf{X}(x)Y(y) \tag{15}$$

where $\mathbf{X}(x)=[W(x),\theta(x),T(x),M_y(x)]^T$. Substituting Eq. (15) into Eq. (14), we find

$$\frac{dY(y)}{dy}=\mu Y(y);\quad \mathbf{HX}(x)=\mu\mathbf{X}(x) \tag{16a,b}$$

where μ is the eigenvalue and $\mathbf{X}(x)$ is the corresponding eigenvector.

By expanding Eq. (16b) the eigen solution can be obtained via the ordinary differential equation

$$\frac{d^4W(x)}{dx^4}+2\mu^2\frac{d^2W(x)}{dx^2}+\mu^4W(x)=0 \tag{17}$$

The solution of Eq. (17) is

$$W(x)=A\cos\mu x+B\sin\mu x+Cx\cos\mu x+Fx\sin\mu x \tag{18}$$

For a plate slidingly supported at $x=0$ and simply supported at $x=a$, the boundary conditions are

$$\left.\frac{\partial W(x)}{\partial x}\right|_{x=0}=V_x(x)\big|_{x=0}=0;\quad W(x)\big|_{x=a}=M_x(x)\big|_{x=a}=0 \tag{19}$$

Substituting Eq. (18) into Eq. (19) then equating the determinant of the coefficient matrix to zero, we have the transcendental equation of eigenvalues

$$\cos^2(\mu a)=0 \tag{20}$$

with the double roots

$$\mu_n=\frac{n\pi}{2a};\quad \mu_{-n}=\frac{-n\pi}{2a}\quad (n=1,3,5,\ldots) \tag{21}$$

The corresponding basic eigenvector of μ_n, obtained from Eq. (16b), is

$$\mathbf{X}_n^0(x)=[1,\mu_n,D\mu_n^3(\nu-1),D\mu_n^2(\nu-1)]^T\cos(\mu_n x) \tag{22}$$

Knowing that the eigenvalue μ_n is a double root, there exists the first-order Jordan form eigen solution $\mathbf{X}_n^1$, which is solved by $\mathbf{HX}_n^1=\mu_n\mathbf{X}_n^1+\mathbf{X}_n^0$ while imposing the boundary conditions (19), as

$$\mathbf{X}_n^1(x)=\left[1,\ 1+\mu_n,\ D\mu_n^2(\mu_n\nu-\mu_n+\nu+1),\ D\mu_n(\mu_n\nu-\mu_n-2)\right]^T\cos(\mu_n x) \tag{23}$$

In a similar manner, the corresponding basic and first-order Jordan form eigenvectors for the eigenvalue $-\mu_n$ are

$$\mathbf{X}_{-n}^0(x)=[1,-\mu_n,-D\mu_n^3(\nu-1),D\mu_n^2(\nu-1)]^T\cos(\mu_n x) \tag{24}$$