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Vibration of a variable cross-section beam

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Abstract

Vibration of an isotropic beam which has a variable cross-section is investigated. Governing equation is reduced to an ordinary differential equation in spatial coordinate for a family of cross-section geometries with exponentially varying width. Analytical solutions of the vibration of the beam are obtained for three different types of boundary conditions associated with simply supported, clamped and free ends. Natural frequencies and mode shapes are determined for each set of boundary conditions. Results show that the non-uniformity in the cross-section influences the natural frequencies and the mode shapes. Amplitude of vibrations is increased for widening beams while it is decreased for narrowing beams.

Keywords: Beam; Variable cross-section; Vibration; Analytical solution

1. Introduction

Beams are used as structural component in many engineering applications and a large number of studies can be found in literature about transverse vibration of uniform isotropic beams (Gorman, 1975). Non-uniform beams may provide a better or more suitable distribution of mass and strength than uniform beams and therefore can meet special functional requirements in architecture, robotics, aeronautics and other innovative engineering applications and they have been the subject of numerous studies. Cranch and Adler (1956) presented the closed-form solutions (in terms of the Bessel functions and/or power series) for the natural frequencies and mode shapes of the unconstrained non-uniform beams with four kinds of rectangular cross-sections. Similar closed-form solutions for the truncated-cone beams and the truncated-wedge beams were obtained by Conway and Dubil (1965). Heidebrecht (1967) determined the approximate natural frequencies and mode shapes of a non-uniform simply supported beam from the frequency equation using a Fourier sine series. Branch (1968) optimized fundamental frequency of transverse oscillation of beams with variable cross-section which is allowed to vary in a manner such that the second area moment is linearly related to the area. Mabie and Rogers (1972) considered polynomial variation of the beam cross-sectional area and the moment of inertia and obtained natural frequencies for a double-tapered beam. Bailey (1978) numerically solved the frequency equation derived from the Hamilton's law to obtain the natural frequencies of the non-uniform cantilever beams.

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Olhoff and Parbery (1984) used cross-sectional area function as the design variable to maximize the difference between two adjacent natural frequencies. Gupta (1985) numerically determined the natural frequencies and mode shapes of the tapered beams using a finite element method. Jategaonkar and Chehil (1989) studied non-uniform beams with cross-section varying in a continuous or non-continuous manner along their lengths. Naguleswaran (1992, 1994a) determined the approximate natural frequencies of the single-tapered beams and double-tapered beams with a direct solution of the mode shape equation based on the Frobenius method. Naguleswaran (1994b) also investigated a uniform beam of rectangular cross-section one side of which varies as the square root of the axial coordinate. Laura et al. (1996) used approximate numerical approaches to determine the natural frequencies of Bernoulli beams with constant width and bilinearly varying thickness. Datta and Sil (1996) numerically determined the natural frequencies of cantilever beams with constant width and linearly varying depth. Caruntu (2000) examined the nonlinear vibrations of beams with rectangular cross section and parabolic thickness variation. Recently, Elishakoff and Johnson (2005) investigated the vibration problem of a beam which has axially non-uniform material properties. Free vibration of stepped beams has also received a considerable attention and a comprehensive review is given by Jang and Bert (1989a,b). Some of these results can also be found in the monograph by Elishakoff (2005).

Previous studies clearly show that vibration characteristics of isotropic beams with continuously changing cross-section have significant features and are not yet fully addressed. The present study investigates free vibration of an isotropic beam with exponentially varying width. The object is to obtain analytical solutions describing the vibration behavior of the beam under different boundary conditions and to determine the effects of continuously variable cross-section on the natural frequencies and mode shapes.

2. Analysis

Consider an isotropic beam with a variable cross-section. Dimensionless variables are defined according to

$$t=\frac{1}{L^2}\sqrt{\frac{EI_0^*}{\rho A_0^*}}t^*,\quad x=\frac{x^*}{L},\quad I=\frac{I^*}{I_0^*},\quad w=\frac{w^*}{W},\quad A=\frac{A^*}{A_0^*},\tag{1}$$

where t^* is the dimensional time, x^* is the dimensional coordinate measured from the left end of the beam along its length, A^* and I^* are the dimensional area and moment of inertia of the cross-section of the beam respectively, w^* is the dimensional transverse displacement, ρ is the mass density per unit are of the beam, E is the Young's modulus, L is the length of the beam, W is any reference displacement and A_0^* and I_0^* are respectively the area and moment of inertia of the cross-section of the beam at the left end of the beam where $x=0$ that is $A_0^*=A_0^*(0), I_0^*=I_0^*(0)$. Governing equation in the dimensionless form can be written as follows:

$$\frac{I(x)}{A(x)}\frac{\partial^4 w}{\partial x^4}+2\frac{I'(x)}{A(x)}\frac{\partial^3 w}{\partial x^3}+\frac{I''(x)}{A(x)}\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial t^2}=0.\tag{2}$$

Solution of the Eq. (2) can be assumed in the following form:

$$w(x,t)=F(x)G(t).\tag{3}$$

Substitution of Eq. (3) into Eq. (2) yields two ordinary differential equations.

$$\frac{I(x)}{A(x)}F^{(4)}+2\frac{I'(x)}{A(x)}F'''+\frac{I''(x)}{A(x)}F''-\omega^2F=0,\tag{4}$$

$$G''+\omega^2G=0.\tag{5}$$

Here ω is a real constant and defined as $\omega^2=(\Omega^2\rho L^4/EI_0)$ and Ω is radial frequency. Solution of Eq. (5) is well known and can be written as

$$G(t)=C_1\cos(\omega t)+C_2\sin(\omega t).\tag{6}$$

Solution of Eq. (4) requires the geometry of the cross-section of the beam to be specified. A cross-section geometry for which both the area and the moment of inertia are directly proportional to the characteristic width is considered in the present study. The family of the cross-section geometries possessing these features