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Two theorems on the steady-state conduction in anisotropic body

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Abstract

In this paper, steady-state plane conduction problems are considered. Two theorems are proven on the conductance of non-homogeneous and anisotropic plane conductors. Here, the conductor is a two-dimensional bounded plane domain with a conducting matter having two seperated boundary terminals and two seperated insulated boundary segments. Known theorems referring to homogeneous and isotropic conductors have been extended to non-homogeneous and anisotropic ones. The results of the paper can be directly used in following fields: heat flow, diffusion, irrotational hydraulic flow, flow of electrical current and anti-plane shear deformation. Application of formulae derived is illustrated in the examples of heat flow and anti-plane elastic shear deformation.

Keywords: Anisotropic; Anti-plane shear deformation; Conductor; Heat flow; Non-homogeneous; Steady-state

1. Introduction

This paper deals with the conductance between the terminal surfaces attached to a conducting body. The steady-state two-dimensional conduction problems are considered. In the present paper the conducting body (conductor) is a two-dimensional simply connected bounded (x, y) plane region A with boundary arcs $\partial A_1 = \widehat{P_1P_2}$, $\partial A_2 = \widehat{P_3P_4}$ as terminals. The complementary part of boundary curve of A consists of arcs $\partial A_3 = \widehat{P_2P_3}$ and $\partial A_4 = \widehat{P_4P_1}$ which are insulated. The boundary segments ∂A_1 and ∂A_2 are separated by the boundary segments ∂A_3 and ∂A_4 as shown in Fig. 1. The whole boundary curve of A is $\partial A = \bigcup_{i=1}^{4} \partial A_i$. The theorems proved are known for uniform homogeneous and isotropic conductors (Duffin, 1959; Rayleigh, 1876). Here, the non-homogenity and anisotropy of the conductor are also considered.

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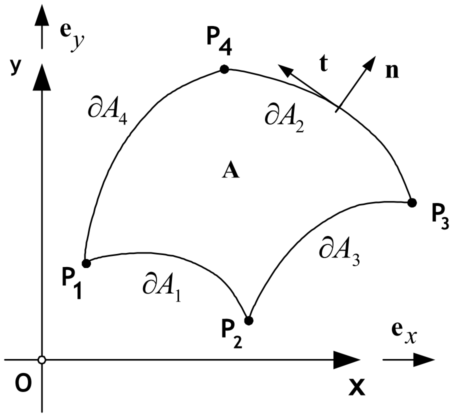

Fig. 1. Plane conductor.

The anisotropy and non-homogenity of the material properties of the two-dimensional conductors are described by the tensor field $\mathbf{K} = \mathbf{K}(x, y)$. $\mathbf{K}$ is a two-dimensional second-order uniformly positive definite, bounded, symmetric tensor field on $\bar{A} = A \cup \partial A$ and $\mathbf{K} \in C^2(\bar{A})$. $\mathbf{K}$ has the form

$$\mathbf{K} = a(x,y)\mathbf{e}_x \circ \mathbf{e}_x + c(x,y)(\mathbf{e}_x \circ \mathbf{e}_y + \mathbf{e}_y \circ \mathbf{e}_x) + b(x,y)\mathbf{e}_y \circ \mathbf{e}_y, \tag{1}$$

where $a > 0$, $b > 0$, $ab - c^2 > 0$ in $\bar{A}, a, b, c \in C^2(\bar{A})$, $\mathbf{e}_x$, $\mathbf{e}_y$ are the unit vectors of the coordinate system Oxy and the circle between two vectors denotes their tensorial product (Lurje, 1970; Malvern, 1969). Here, we note a, b, c are units free.

The following boundary value problem arises in connection with the steady-state in-plane conductance:

$$\nabla \cdot (\mathbf{K} \cdot \nabla u) = 0 \quad \text{in } A, \tag{2}$$

$$u = 0 \quad \text{on } \partial A_1, \qquad u = 1 \quad \text{on } \partial A_2, \tag{3}$$

$$\mathbf{n} \cdot \mathbf{K} \cdot \nabla u = 0, \quad \text{on } \partial A_3 \cup \partial A_4 \tag{4}$$

In Eqs. (2)–(4)

$\nabla = \frac{\partial}{\partial x}\mathbf{e}_x + \frac{\partial}{\partial y}\mathbf{e}_y$ is the gradient (del) operator (Lurje, 1970; Malvern, 1969),

$\mathbf{n} = n_x\mathbf{e}_x + n_y\mathbf{e}_y$ is the outer unit normal vector of curve ∂A,

dot denotes the scalar product according to Lurje (1970) and Malvern (1969). We note, $u = u(x, y)$ is also units free.

Following Mikhlin (1964) we can associate the next energy integral to the boundary value problem formulated by Eqs. (2)–(4)

$$E_K[u] = \int_A \nabla u \cdot \mathbf{K} \cdot \nabla u \, \mathrm{d}A. \tag{5}$$

It is clear, $E_K[u] > 0$ (Bergman and Schiffer, 1953; Mikhlin, 1964). We can check

$$E_K[u] = -\int_{\partial A_1} \mathbf{n} \cdot \mathbf{K} \cdot \nabla u \mathrm{d}s = \int_{\partial A_2} \mathbf{n} \cdot \mathbf{K} \cdot \nabla u \, \mathrm{d}s, \tag{6}$$

where s is an arc coordinate defined on the boundary curve ∂A. The validity of Eq. (6) follows from:

$$\int_A \nabla u \cdot \mathbf{K} \cdot \nabla u \, \mathrm{d}A = \int_A \nabla \cdot (u\mathbf{K} \cdot \nabla u) \, \mathrm{d}A - \int_A u \nabla \cdot (\mathbf{K} \cdot \nabla u) \, \mathrm{d}A = \int_{\partial A} u\mathbf{n} \cdot \mathbf{K} \cdot \nabla u \, \mathrm{d}s = \int_{\partial A_2} \mathbf{n} \cdot \mathbf{K} \cdot \nabla u \, \mathrm{d}s \tag{7}$$