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Analytic solution for a quartic electron mirror

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ABSTRACT

A converging electron mirror can be used to compensate for spherical and chromatic aberrations in an electron microscope. This paper presents an analytical solution to a diode (two-electrode) electrostatic mirror including the next term beyond the known hyperbolic shape. The latter is a solution of the Laplace equation to second order in the variables perpendicular to and along the mirror's radius ($z^2-r^2/2$) to which we add a quartic term ($k\lambda z^4$). The analytical solution is found in terms of Jacobi cosine-amplitude functions. We find that a mirror less concave than the hyperbolic profile is more sensitive to changes in mirror voltages and the contrary holds for the mirror more concave than the hyperbolic profile.

1. Introduction

In 1990 Gertrude F. Rempfer [1] built on the research of those such as Zworykin et al. [2] and Ramberg [3] to lay the theoretical foundations for the hyperbolic electron mirror as a means to counter the spherical and chromatic aberrations of electron lenses since aberrations in the former are of opposite sign to those in the latter. A number of other researchers have also employed such mirrors [4–8].

In order to gain more flexibility in the ratio of these two aberrations Shao and Wu [9] designed a four-element mirror whose outer elements are not hyperbolic. Likewise, Fitzgerald, Word, and Könenkamp [10] extended Rempfer's diode hyperbolic electron mirror to include a third hyperbolic electrode that provides more flexibility in the choice of potentials to better match spherical and chromatic aberrations and allow for different magnifications.

The present paper returns to the diode case to examine the utility of a mirror whose profile deviates from a hyperbolic surface, either more concave or more convex, in order to better describe the equipotentials that result from the finite radii of real mirrors. As an added benefit, we find that mirrors that are more concave than the hyperbolic profile can provide aberration corrections that are more stable against fluctuations in the applied electric field. Given that voltages inevitably fluctuate, this robustness will likely improve the resolution of images.

2. Theoretical model of a quartic diode mirror

Our approach to the more general class of mirror profiles closely tracks the case of the hyperbolic diode mirror [1]. At each step we reduce the extended results to the known hyperbolic profile that Rempfer found.

The one exception to that flow is to bypass the derivation of solutions to Laplace's equation for the potential in cylindrical coordinates in terms of the axial potential $V(z)$ near the axis that Rempfer [1] and Shao and Wu [9] use. Instead we simply write down the most general solution [11] in terms of the radial r, axial z, and rotational ϕ variables, as well as the coordinate-separation constants β and m.

$$V(r,z,\phi)=\begin{Bmatrix} J_m(\beta r) \\ Y_m(\beta r)\end{Bmatrix}\begin{Bmatrix} e^{\beta z} \\ e^{-\beta z}\end{Bmatrix}\begin{Bmatrix} e^{im\phi} \\ e^{-im\phi}\end{Bmatrix}. \tag{1}$$

(Here the stacked brackets are shorthand for the sum of eight possible products of three factors.) For rotationally symmetric solutions, $m\equiv 0$. Since we will place an electrode at some finite potential passing through $z=0$, as in Fig. 1, we exclude the Bessel function of the second kind, $Y_0(\beta r)$, that diverges there. Finally, for V nonzero at the origin (since we wish to reverse the path of electrons with a positively-charged electrode at the origin) we have the symmetric sum of exponentials,

$$V(r,z)=V_M\cosh(\beta z)J_0(\beta r). \tag{2}$$

The functions cosh and J_0 can each be replaced with a series representation,

$$V(r,z)-V_M=V_M\left\{1+\frac{\beta^2z^2}{2}+\frac{\beta^4z^4}{4!}+\cdots\right\}\times\left\{1-\frac{\beta^2r^2}{2^2}+\frac{\beta^4r^4}{2^4(2!)^2}-\cdots\right\}-V_M=$$

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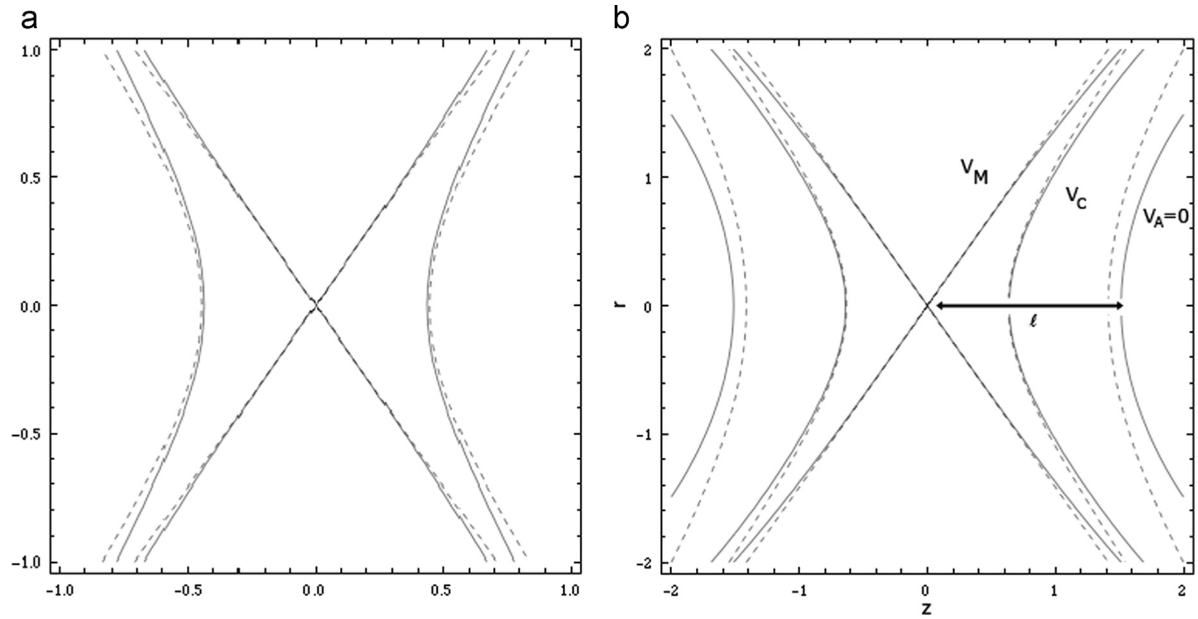

Fig. 1. Theoretical model of the quartic diode electron mirror. Dashed lines are the potential surfaces for the hyperbolic mirror profile, having cylindrical symmetry about the z-axis with $z=0$ at the cone vertex. (a) Solid lines show the deviation of two quartic potential surfaces to a profile more convex than the hyperbolic case, like the bell of a trumpet. (b) Solid lines show the deviation of the quartic diode mirror to a profile more concave than the hyperbolic diode mirror, like a tulip blossom. The voltages V_M and V_A are on the mirror and terminating (here, grounded) electrodes, respectively. The distance from the vertex to the opening of the aperture is the mirror length ℓ, which we note has a larger value of z than in the hyperbolic diode mirror. A small aperture in the terminating electrode allows electron to enter and exit the mirror field.

$$k\left\{z^2-\frac{r^2}{2}\right\}+\frac{k^2}{2V_M}\left\{\frac{z^4}{3}+\frac{r^4}{2^3}-r^2z^2\right\}+\cdots, \tag{3}$$

where we have subtracted the particular solution $V_0=V_M$ – the potential at the origin – from both sides and where $k=(\beta^2/2)V_M$. The first term in the last expression is the potential between the electrodes in a rotationally symmetric hyperboloid field, the same arrived at via expansion of V in a conventional power series in r [12]. An analytic solution for the trajectory of an electron in this potential exists, from which the paraxial object/image distance and chromatic and spherical aberration coefficients have been derived [1].

Since r is generally close to the axis, the next term in the last expression, proportional to $(k^2/2V_M)z^4/3$, is the next largest contributor. We here undertake to find the analytical solution for electron trajectories in a potential containing this term, too.

The present problem, then, is to find the equations of motion for an electron in such a potential and to solve these equations for the position of the electron as a function of time. Fig. 1a shows a cross section of a pair of equipotential surfaces for the hyperbolic case (dashed lines) and the flaring of such equipotentials outward from the horizontal axis with the introduction of the quartic term $(k^2/2V_M)z^4/3$ in the potential (solid lines).

If, on the other hand, we change the sign of that quartic term, the equipotential surfaces will contract inward toward the horizontal axis as in Fig. 1b. Fig. 1b shows an equipotential labeled V_A that is the physical conformation of a grounded electrode in a physical mirror, one containing a small aperture to let electrons pass through from the right. The equipotential labeled V_M would also be the conformation of a physical electrode held at a negative voltage to stop the electron at equipotential V_C (which is not a physical electrode) and reverse its course.

One would suppose that this inward contraction seen in Fig. 1b would tend to more strongly focus the electron in its return trajectory, while an equivalent mirror based on the outward flaring of Fig. 1a would tend to reduce the focusing of the mirror. Our goal in the next section is to turn these suppositions into precise analytical trajectories.

2.1. Electron trajectories in a quartic field, general solution

The equations of motion we need to solve, for an electron in a quartic potential field, are

$$\frac{d^2r}{dt^2}=\frac{e}{m}\frac{\partial V}{\partial r}=-\frac{ek}{m}r=-\omega^2 r, \tag{4a}$$

$$\frac{d^2z}{dt^2}=\frac{e}{m}\frac{\partial V}{\partial z}=2\omega^2 z+2\omega^4\mu z^3, \tag{4b}$$

where e and m are the charge and mass of an electron and $\omega=\sqrt{ek/m}$. In the following $\mu=m/(3eV_M)$ will be used as a quartic perturbation parameter that may be set to 0 to recover the hyperbolic case in the expressions below.

Although Eq. (4b) is a nonlinear second-order differential equation, we suspected that it might be solved using Jacobi elliptic functions [13], since the *Jacobi cosine-amplitude* $cn(u|\kappa)$, for instance, is a solution [14] to the differential equation

$$\frac{d^2y}{dt^2}=(2-k^2)y-2y^3. \tag{5}$$

Under that supposition the solution would be of the form

$$z(t)=A\,\mathrm{cn}(B(\sqrt{2}\;t\omega-\psi))|\kappa), \tag{6}$$

where the parameters A, B, ψ, and κ will be determined to satisfy Eq. (4b). The derivative is

$$\dot z(t)=-\sqrt{2}AB\omega\,\mathrm{dn}\left(B\left(\sqrt{2}t\omega-\psi\right)|\kappa\right)\times\mathrm{sn}\left(B\left(\sqrt{2}t\omega-\psi\right)|\kappa\right), \tag{7}$$

which is expressed in terms of Jacobi delta-amplitude $\mathrm{dn}(u|\kappa)$ and sine-amplitude $\mathrm{sn}(u|\kappa)$ functions. We assert the boundary condition that the electron location at $t=0$ is at the potential surface where $\dot z$ and $\dot r$ are both zero, the point of furthest penetration into the quartic field (V_C in Fig. 1b).

At this point the incident and return electron trajectories are perpendicular to the reflecting potential surface. Since $\mathrm{sn}(0|\kappa)=0$, [15] then $\dot z(0)=0$ requires $\psi\equiv 0$.