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Efficient solution of the first passage problem by Path Integration for normal and Poissonian white noise

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ABSTRACT

In this paper the first passage problem is examined for linear and nonlinear systems driven by Poissonian and normal white noise input. The problem is handled step-by-step accounting for the Markov properties of the response process and then by Chapman–Kolmogorov equation. The final formulation consists just of a sequence of matrix–vector multiplications giving the reliability density function at any time instant. Comparison with Monte Carlo simulation reveals the excellent accuracy of the proposed method.

1. Introduction

The first passage problem has been investigated in many publications over almost a century because of its relationship to the safety of structural systems under random excitations. The exact solution to the first passage problem is not available because even in the case of a normal white noise process the Fokker–Planck equation with associated boundary conditions is in general unknown [1]. Many approximate methods have been proposed [2–7], however, the analytical approximation methods are available only for light damping for the stochastic averaging, weak nonlinearity and Gaussian approximations. First passage time for linear systems with stochastic coefficients has been addresses by using Pontryagin–Vitt equations in [8]. A quite different approach for non-linear equations driven by normal white noise has been proposed in [9] by using a generalized cell mapping method. Other relevant contributions on the subject may be found in [10–12].

In order to determine the probability distribution of the first passage time, efficient solution of the Fokker–Planck equation is necessary. Moreover, we need a solution of the problem step-by-step in order to cancel the trajectories that for the first time leave the safe domain (absorbing barrier problem). In order to have such a control on the path of the trajectories the only way is using the so-called *Path Integration* (PI) method. It mainly consists of using the Chapman–Kolmogorov (CK) equation giving the probability density at a certain time instant as weighted sum of the contributions of the various trajectories that in a previous time instant start with deterministic initial condition. As the interval between the two time instants becomes small, then the so-called *short time Gaussian approximation* [13] remains still valid and the step-by-step solution technique of the CK equation reverts to the PI method. Many papers have been devoted to this subject for normal [14–19] and Poissonian white noise as well as renewal processes [20–23].

The PI method is versatile and in [24] it has been used for solving the first passage problem. It mainly consists in defining the so-called *reliability function* which is a function giving the probability that the various trajectories will remain inside the safe barrier conditioned by the fact that each of them never crosses the barrier up to the observation time.

In this paper, by using the concepts exploited in [24] the first passage problem is revisited in the light of the cell mapping method and extended to the case of Poissonian white noise input. It is shown that the reliability function by discretization of the Chapman–Kolmogorov equation may be easily implemented in a computer program as just a sequence of matrix–vector multiplications whose sizes depend on the threshold barriers and the spatial discretization steps. Moreover, as the input is stationary the reliability function is governed by a transition matrix that does not explicitly depend on time so that it can be computed once

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beforehand.

The paper is organized as follows. In Section 2 the PI, some well-known concepts of the PI method for both normal and Poissonian white noise are presented for clarity's sake as well as for introducing appropriate notations. These concepts are framed in the context of a cell-mapping method. In Section 3 the first passage time by using PI method is presented for both normal and Poissonian white noise for the half oscillator. In Section 4 the extension to a single degree-of-freedom oscillator is presented while in Section 5 the numerical applications are presented and the results are compared with those obtained from Monte-Carlo simulation.

2. Path Integration method

In this section some preliminary concepts on Path Integral Solution (PIS) will be introduced for clarity's sake as well as for introducing appropriate notation. Let any nonlinear system be governed by the equation

$$\dot{X} + f(X, t) = W(t) \tag{1}$$

where $f(X, t)$ is any non-linear function of the response process $X(t)$ and $W(t)$ is a normal white noise characterized by the strength Q. This means that

$$E[W(t_1)W(t_2)] = Q\delta(t_1 - t_2) \tag{2}$$

where $E[\cdot]$ denotes ensemble average and $\delta(\cdot)$ is Dirac's delta function. The response process $X(t)$ is Markovian and the Chapman–Kolmogorov equation

$$p_X(x, t+\tau) = \int_{-\infty}^{\infty} p_X(x, t+\tau|y, t)p_X(y, t)\,\mathrm{d}y \tag{3}$$

holds true. In Eq. (3) $p_X(x, t)$ is the probability density function (PDF) of the process $X(t)$ at time t and $p_X(x, t+\tau|y, t)$ is the conditional PDF at time $t+\tau$ for an assigned (deterministic) initial condition y at time t. Eq. (3) is valid for any value of τ. However, for finding the evolution in time of the PDF of the response process $X(t)$, the Chapman–Kolmogorov equation is written for a small value of τ that will be denoted as $\tau = \Delta t$, and Eq. (3) particularized for $\tau = \Delta t$ is usually called *Path Integral* (PI).

2.1. Gaussian white noise

In this case the conditional PDF in Eq. (3) is determined from the so-called *short time Gaussian approximation* [13]. A deeper insight into the concept is necessary in order to clearly understand the use of Eq. (3) particularized for $\tau = \Delta t$. The conditional PDF in Eq. (3) is the solution to the Fokker–Planck (FP) equation associated to Eq. (1) with the assigned deterministic initial condition $x(t_0) = y$ in t_0. It is obvious that if we know the transient solution of the FP equation for any value of y we may also solve the FP equation for the original system (1). In order to get the conditional PDF in Eq. (3) we subdivide the t-axis into small intervals of equal length Δt and rewrite this equation into the form

$$p_X(x, t_k + \Delta_t) = \int_{-\infty}^{\infty} p_X(x, t_k + \Delta t|y, t)p_X(y, t_k)\,\mathrm{d}y \tag{4}$$

Then we define a new process $\bar{X}(\tau)$ governed by the equation

$$\dot{\bar{X}} + f(\bar{X}(\tau), \tau) = W(t_k + \tau); \quad \bar{X}(0) = y \tag{5}$$

This situation is depicted in Fig. 1.

Now in virtue of the short time Gaussian approximation since

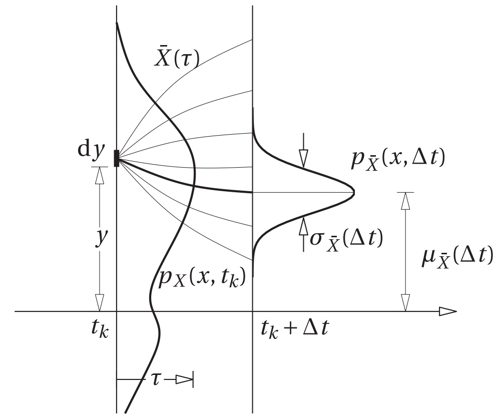

Fig. 1. Trajectories and PDF of the process $\bar{X}(\tau)$.

$$E[\bar{X}(\Delta t)] = \mu_{\bar{X}}(\Delta t) \approx y - f(y, t_k)\Delta t$$
$$\sigma^2_{\bar{X}}(\Delta t) \approx Q\Delta t \tag{6}$$

as we assume that the PDF of the process $\bar{X}(\tau)$ in the interval $0 \le \tau \le \Delta t$ is Gaussian, then

$$p_{\bar{X}}(x, \Delta t) = p_X(x, t_k + \Delta t|y, t_k) = \frac{1}{\sqrt{2\pi}\sigma_{\bar{X}}(\Delta t)} \exp\left(-\frac{\left(x - \mu_{\bar{X}}(\Delta t)\right)^2}{2\sigma^2_{\bar{X}}(\Delta t)}\right) \tag{7}$$

By inserting Eq. (7) into Eq. (4) the step-by-step-solution may be readily found.

If the system is linear, namely $f(X, t) = aX$ $(a > 0)$, then the exact values of the mean and the variance of the process $\bar{X}(t)$ is readily found in the form

$$\mu_{\bar{X}}(\Delta t) = y\exp(-a\Delta t); \quad \sigma^2_{\bar{X}}(\Delta t) = \frac{Q}{2a}\left(1 - \exp(-2a\Delta t)\right) \tag{8}$$

Now we may give an interpretation of Eq. (4) that will be useful for the first passage problem. We have the process $X(t)$, solution of Eq. (1) from the whole sample functions, some of them lie within the interval $[y, y+\mathrm{d}y]$, and this happens with probability $p_X(y, t)\,\mathrm{d}y$. These trajectories generate the process $\bar{X}(t)$ that is characterized in $t_k + \Delta t$ by the conditional PDF given in Eq. (7), then Eq. (4) gives $p_X(x, t_k + \Delta t)$ as the sum of the contribution of $p_{\bar{X}}(x, \Delta t)$ weighted by $p_X(x, t_k)$ (see Fig. 1). This perspective is important for the first passage problem since, as the cell mapping method [9] we have control on the various trajectories.

2.2. Poissonian white noise

For the case of Poisson white noise the PI has been performed in [17]. Herein this is briefly summarized. Let the equation of motion (1) be driven by a Poisson white noise $W_p(t)$. It is defined as

$$W_p(t) = \sum_{k=1}^{N(t)} z_k\delta(t - t_k) \tag{9}$$

where z_k is the k-th realization of a random variable Z with assigned probability density function $P_Z(z)$, t_k is the k-th realization of a random variable T distributed in time according to the Poisson law with expected arrival rate λ and $N(t)$ is the number of spikes within the interval $[0, t]$. A sample function $W_p(t)$ of such a process is depicted in Fig. 2(a).

By integrating Eq. (1) for each sample function we get the