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An asymptotic expansion-based method for a spectral approach in equivalent statistical linearization

T. Canor [a,b,*], N. Blaise [a], V. Denoël [a]

[a] University of Liège, Department of Architecture, Geology, Environment and Constructions, Structural Engineering Division, Chemin des Chevreuils, 1, B52/3, 4000 Liège, Belgium
[b] F.R.S.-FNRS, National Fund for Scientific Research, 1000 Bruxelles, Belgium

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ABSTRACT

Equivalent linearization consists in replacing a nonlinear system with an equivalent linear one whose parameters are tuned with regard to the minimization of a suitable function. In particular, the Gaussian equivalent linearization expresses the properties of an equivalent linear system in terms of the mean vector and the covariance matrix of the responses, which are the unknowns of the optimization problem in a spectral approach. Even though the system has been linearized, the resulting set of equations is nonlinear. The computational effort in this method pertains to the solution of a possibly large set of nonlinear algebraic equations involving integrals and inversions of full matrices. This work proposes to develop and apply an asymptotic expansion-based method to facilitate and to improve the statistical linearization for large nonlinear structures. The proposed developments demonstrate that for slightly to moderately coupled nonlinear systems, the equivalent linearization can be applied with an appropriate modal approach and eventually seen as a convergent series initiated with the stochastic response of a main decoupled linear system. With this method, the computational effort is attractively reduced, the conditioning of the set of nonlinear algebraic equations is improved and inversion of full transfer matrices and repeated integrations are avoided. The paper gives a formal description of the method and illustrates its implementation and performances with the computation of stationary responses of nonlinear structures subject to coherent random excitation fields.

1. Introduction

A classical result from probabilistic theory states that linear deterministic systems driven by Gaussian processes respond with Gaussian processes. The joint probability density function of the responses is thus completely characterized by a mean vector and a covariance matrix. However, for nonlinear systems or in the case of non-Gaussian excitations, the computation of the response of the system is more complicated, partly due to the statistical polymorphism of a non-Gaussian process.

Discarding straightaway the Fokker–Planck equation due to the curse of dimensionality [1,2], the Monte Carlo approach is considered as the only tractable method to compute the non-Gaussian response of large-dimensional nonlinear systems [3,4]. Briefly, the method consists in generating samples of the excitation to compute samples of the system response by means of deterministic solvers. Although the simulation-based framework is extensively used in risk analysis and risk quantification [5], the computational burden remains a major drawback of this method. Indeed, the generation of random samples from coherent random fields, as wind acting on large structures [6,7], may be prohibitive. Therefore, the use of approximate methods is attractive, especially in a design stage or in an optimization procedure involving many parameters and repeated operations.

Many approximate methods have been developed for decades: the averaging method [8], the equivalent linearization [9,10], quadratization and cubicization [11], non-Gaussian closures [12] are seemingly the most famous. Among them, the equivalent linearization, originally introduced by Botoon and Caughey [13,14], can be used for the analysis of high-dimensional nonlinear structures, as encountered in earthquake engineering [15–17] or in wind engineering [18–20]. The main idea of the equivalent linearization consists in replacing the nonlinear system with an equivalent linear one by minimizing an error criterion depending on the parameters of the equivalent system. Though different criteria have been proposed [9,21], the most robust and advantageous one remains the minimization of the mean squared discrepancy, by tuning the parameters of the equivalent system, especially as the excitations are diffusive processes.

* Corresponding author at: University of Liège, Department of Architecture, Geology, Environment and Constructions, Structural Engineering Division, Chemin des Chevreuils, 1, B52/3, 4000 Liège, Belgium.
E-mail addresses: t.canor@ulg.ac.be (T. Canor), n.blaise@ulg.ac.be (N. Blaise), v.denoel@ulg.ac.be (V. Denoël).

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The equivalent linearization method benefits from valuable features of linear systems. First, the modal projection can be used to reduce the size of the equation of motion. Second, it is more convenient to work in the frequency domain for stationary processes, while realistic loadings are usually modeled by large Power Spectral Density (PSD) matrices, as it is the case for coherent wind or seismic fields. Finally, the input–output Gaussianity is preserved. Consequently, the equivalent linearization may compete with Monte Carlo simulation in the estimation of the first two statistical moments.

In the linearization method, assumptions on the statistical distribution of the response are formulated: the Gaussian Equivalent Linearization (GEL) supposes that the responses of the system are Gaussian processes, but statistical linearization methods have been extended to non-Gaussian processes [22–24] with limited success. The GEL expresses the properties of the equivalent linear system in terms of the mean vector and the covariance matrix of the response of the system. Even though the system has been linearized, the set of equations to calculate the covariance matrix of the system is nonlinear. The computational effort in this method pertains to the solution of a possibly large set of nonlinear algebraic equations, all the more for large nonlinear structures.

This work proposes to develop and apply a perturbation approach, as formerly investigated by the authors for deterministic [25] and stochastic [26] linear systems, to facilitate and to improve the GEL of a nonlinear structure subject to stationary loadings. Our approach exposed in a stationary setting can be extended to some classes of evolutionary problems [27,28] (alternative to fully nonstationary excitations [29]), provided the quasi-stationary assumption is justified, *i.e.* the natural period of the structure is small compared to the duration of the evolutionary random excitation [30,31].

Since many optimization algorithms can be used to solve the nonlinear equation set inherent to statistical linearization, the development of a solver accounting for the specificities of this set has not often been addressed by the research community. Provided the excitation can be modeled by filtered white noises, the covariance matrix of the response is expressed by a Lyapunov equation. This equation can be solved in particular by direct algorithm [32–34]. Nonetheless, if the excitation is modeled as a coherent field in the frequency domain, the Itô procedure cannot be applied. In the context of equivalent linearization with a spectral approach, a fixed-point algorithm is a convenient and readily implemented method [10]. However, this algorithm behaves poorly in terms of convergence [35], especially for large equation sets. Consequently, a gradient-based formulation is preferred to circumvent some limitations.

The proposed developments demonstrate that for slightly to moderately coupled nonlinear systems in a suitable modal basis, the equivalent linearized response can be seen as a convergent series of correction terms initiated with the stochastic response of a main decoupled linear system. This work shows that the concept of asymptotic expansion of a modal transfer matrix might be efficiently used to enhance the GEL technique. Indeed, this expansion allows us to compute rapidly the Jacobian matrix required in a gradient-based method. The conditioning of the system is also improved, especially for large structures. The computational effort is thus attractively reduced, while preserving the advantages of spectral analysis.

Because simulation techniques or alternative exact approaches would equally perform in small-size structures with simple loadings, a specific attention in the developments is dedicated to high-dimensional structures subject to coherent random excitation fields such as those encountered in wind and earthquake engineering. The proposed method is capable of dealing with nonlinear conservative as well as dissipative forces, either affecting some degrees of freedom only, or more regularly distributed in the whole structure.

First, the philosophy of the equivalent linearization is exposed, then the asymptotic expansion of a modal transfer matrix is developed in the context of GEL. A Newton–Raphson procedure applying the asymptotic expansion is then described. Finally, illustrated examples are proposed to emphasize the pertinence of the method and to highlight the underlying assumptions.

2. Spectral strategy for stochastic linearization of large structures

On a probability space $(\Theta, \mathfrak{F}, \mathbb{P})$, the equation of motion of an n-DOF nonlinear system is

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{C}\dot{\mathbf{y}} + \mathbf{K}\mathbf{y} + \mathbf{g}(\mathbf{y}, \dot{\mathbf{y}}) = \mathbf{f}, \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the $n \times n$-dimensional mass, damping and stiffness matrices of the structure, respectively, $\mathbf{f}(t, \theta) : \mathbb{R}^+ \times \Theta \mapsto \mathbb{R}^n$ is the vector of the random exogenous Gaussian forces and the dot denotes the time derivative. The vector $\mathbf{y}(t, \theta) : \mathbb{R}^+ \times \Theta \mapsto \mathbb{R}^n$ gathers the nodal displacements expected to be non-Gaussian processes due to the nonlinear forces gathered in the vector function $\mathbf{g}(\mathbf{y}, \dot{\mathbf{y}}) : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^n$. With this formalism, the equation of motion is split into four contributions: inertial forces, internal linear forces, internal nonlinear forces and exogenous random forces. Actually, in these developments, we consider nonlinear conservative or dissipative forces only depending on the nodal displacements of the structure (no history variable). Discarding the nonlinear forces $\mathbf{g}(\mathbf{y}, \dot{\mathbf{y}})$ in (1) produces a linear governing equation, referred to as the *linear subsystem* in the sequel.

In the considered problem, the size n of the system is possibly large, and the exogenous forces are characterized by a PSD matrix $\mathbf{S_f}(\omega)$ with possibly complex expressions as typically encountered in realistic wind turbulence model [36] or spatial coherence in seismic engineering [37]. For the sake of clarity in the following analytical developments, only antisymmetric nonlinear forces and zero-mean excitation processes are considered. Otherwise, some minor modifications to the method must be operated to take into account the mean response and the non-centered statistical moments [10].

The stochastic linearization aims at replacing Eq. (1) by the equation of motion of a n-DOF equivalent linear structure. The equivalent equation of motion reads

$$\mathbf{M}\ddot{\mathbf{x}} + (\mathbf{C} + \mathbf{C}_{eq})\dot{\mathbf{x}} + (\mathbf{K} + \mathbf{K}_{eq})\mathbf{x} = \mathbf{f}, \tag{2}$$

where $\mathbf{x}$ and $\dot{\mathbf{x}}$ are the Gaussian nodal displacements and velocities of the equivalent linear system, respectively, and with $\mathbf{K}_{eq}$ and $\mathbf{C}_{eq}$ the equivalent stiffness and damping matrices, respectively [10]. The probabilistic response of the system is thus completely characterized by the symmetric covariance matrices $\boldsymbol{\Sigma}_{\mathbf{x}}$ and $\boldsymbol{\Sigma}_{\dot{\mathbf{x}}}$, obtained by integration of the corresponding PSD matrices:

$$\boldsymbol{\Sigma}_{\mathbf{x}} = \int_{\mathbb{R}} \mathbf{S}_{\mathbf{x}}\, d\omega, \quad \text{and} \quad \boldsymbol{\Sigma}_{\dot{\mathbf{x}}} = \int_{\mathbb{R}} \mathbf{S}_{\dot{\mathbf{x}}}\, d\omega, \tag{3}$$

which are themselves obtained by left- and right-multiplication of $\mathbf{S_f}(\omega)$ by the nodal frequency response function of the system [38].

The equivalent stiffness and damping matrices in (2) are determined by minimizing the error function $\mathcal{E}$ [8], defined as

$$\mathcal{E} = E[(\mathbf{K}_{eq}\mathbf{x} + \mathbf{C}_{eq}\dot{\mathbf{x}} - \mathbf{g}(\mathbf{x}, \dot{\mathbf{x}}))(\mathbf{K}_{eq}\mathbf{x} + \mathbf{C}_{eq}\dot{\mathbf{x}} - \mathbf{g}(\mathbf{x}, \dot{\mathbf{x}}))^T] \tag{4}$$

with $E[\cdot]$ being the expectation operator. Because the covariance between displacements and velocities is equal to zero in a stationary setting $E[\mathbf{x}\dot{\mathbf{x}}^T] = \mathbf{0}$, the equivalent matrices $\mathbf{K}_{eq}$ and $\mathbf{C}_{eq}$