Probabilistic Engineering Mechanics 38 (2014) 35–41

Contents lists available at ScienceDirect

Probabilistic Engineering Mechanics

journal homepage: www.elsevier.com/locate/probengmech

Nonstationary response of nonlinear oscillators with optimal bounded control and broad-band noises

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ARTICLE INFO

Article history:
Received 5 August 2013
Received in revised form
26 July 2014
Accepted 5 August 2014
Available online 17 August 2014

Keywords:
Broad-band excitation
Nonstationary response
Probability density function
Stochastic averaging
Stochastic optimal control

ABSTRACT

Nonstationary response of nonlinear oscillators with optimal bounded control and broad-band noise excitations is investigated. First, the stochastic averaging method is applied to obtain an averaged Itô stochastic differential equation for the amplitude process. Then, the dynamical programming equation is employed to minimize the system response and establish an optimal control law with a control constraint. The nonstationary probability density of the amplitude process can be solved from the corresponding Fokker–Planck–Kolmogorov equation by using the Galerkin method if only external excitations exist. In the case of parametric excitations are present, Monte Carlo simulations can be carried out for the simplified averaged system of the amplitude process with much less computational efforts. Two examples are given to illustrate the feasibility of the proposed procedure and the effectiveness of the optimal control strategy. The accuracy and efficiency of the proposed procedure are substantiated by those obtained from Monte Carlo simulation of the original system.

1. Introduction

Response prediction of mechanical and structural systems subjected to random excitations is important for engineering practice [1,2], and has been extensively studied for several decades. The nonstationary response of a system immediately after exposing to random excitations, also called the transient response, is of importance in reliability analysis. One typical example is a structure in an earthquake, in which case the transient response of the structure during the first several seconds plays a dominant role. Exact solutions for the nonstationary probability density functions of system responses can only be obtained for some linear systems and a few special first-order nonlinear systems [3–5]. For general nonlinear systems, several approximate procedures have been proposed, such as the path integration [6], the cell mapping method [7], and the Galerkin method [8]. In recent years, the Galerkin method and the stochastic averaging method are adopted to explore the nonstationary responses of stochastic nonlinear systems [9–11]. In most of these studies, the random excitations were assumed to be Gaussian white noises due to the ease of mathematical treatment. However, the mathematical treatment is quite complex in the case of non-white random excitations.

If the system response is required to be below a critical level for safety consideration, then a control device with an optimal algorithm may be needed. The theory of stochastic optimal control has been well developed mathematically [12–16]. Two well-known principles for the stochastic optimal control mainly in the fields of economics and finance are Pontryagin's maximum principle and Bellman's dynamical programming. In the engineering field, the linear quadratic Gaussian (LQG) control strategy is widely adopted. Recently, a nonlinear stochastic optimal control strategy was proposed by Zhu and his co-workers [17,18] based on the stochastic dynamical programming [19]. It has been proved to be more efficient than LQG control, and was applied to various nonlinear stochastic systems in the stationary state [20–24]. Combined with the Galerkin method, this control strategy was extended to the nonstationary response of a Rayleigh–Duffing oscillator under external broad-band excitations [25].

In this paper, the procedure developed in [25] is extended to general nonlinear oscillators to control the nonstationary response. The system possesses linear stiffness and nonlinear damping. The random excitations are assumed to be stationary broad-band processes which may be correlated, and they may be external and/or parametric. The stochastic averaging method is applied to the system to obtain an Itô stochastic differential equation of the amplitude envelope. Using the stochastic dynamical programming, the nonstationary optimal control algorithm is applied. If only external excitations are present, the Fokker–Planck–Kolmogorov (FPK) equation governing the nonstationary

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http://dx.doi.org/10.1016/j.probengmech.2014.08.001
0266-8920/

probability density of the amplitude process is then derived, and an approximation procedure of the Galerkin method is carried out to solve this FPK equation. This method is illustrated by a numerical example, and the results obtained are substantiated by those from Monte Carlo simulation (MCS). In the case of parametric excitations existing, MCS can be performed to the averaged one-dimensional system with much less computational time. The second example shows the feasibility of the proposed method.

2. The stochastic model under feedback control

Consider a controlled nonlinear system under random excitations,

$$\ddot{X}+\omega_0^2X+2\alpha\omega_0\dot{X}+f(X,\dot{X})\dot{X}=\sum_{k=1}^{n}g_k(X,\dot{X})\xi_k(t)+u(X,\dot{X}) \tag{1}$$

where $f(X,\dot{X})\dot{X}$ represents the nonlinear damping forces, $u(X,\dot{X})$ denotes a feedback control force, and $\xi_k(t)$ are broad-band stationary random processes with zero means and correlation functions $R_{kr}(\tau)$, namely,

$$R_{kr}(\tau)=E[\xi_k(t)\xi_r(t+\tau)] \tag{2}$$

Each excitation term in (1) could be either external if function $g_k(X,\dot{X})$ is a constant or parametric if $g_k(X,\dot{X})$ depends on X and/or $\dot{X}$. It is assumed that the damping forces, the excitations, and the control force are weak so that the stochastic averaging method is applicable. In more rigorous mathematical terms, each damping term is of an order ε $(0<\varepsilon<1)$, each random excitation term $g_k(X,\dot{X})\xi_k$ is of an order $\varepsilon^{1/2}$, and the feedback control force is of an order ε, so that their contributions to the system response are commensurable.

2.1. Stochastic averaging procedure

Consider the following transformation:

$$X(t)=A(t)\cos\Phi(t) \tag{3a}$$

$$\dot{X}(t)=-A(t)\omega_0\sin\Phi(t) \tag{3b}$$

$$\Phi(t)=\omega_0 t+\Theta(t) \tag{3c}$$

where $A(t)$, $\Phi(t)$, and $\Theta(t)$ are stochastic processes. Using Eqs. (3a)–(3c), the original system equation of motion (1) is transformed to

$$\frac{dA}{dt}=m_1^{(1)}(A,\Theta)+m_1^{(2)}(A,\Theta)+\sum_{k=1}^{n}\sigma_{1k}(A,\Theta)\xi_k(t) \tag{4a}$$

$$\frac{d\Theta}{dt}=m_2^{(1)}(A,\Theta)+m_2^{(2)}(A,\Theta)+\sum_{k=1}^{n}\sigma_{2k}(A,\Theta)\xi_k(t) \tag{4b}$$

where

$$m_1^{(1)}(A,\Theta)=-[2\alpha\omega_0+f(A\cos\Phi,-A\omega_0\sin\Phi)]A\sin^2\Phi \tag{5a}$$

$$m_2^{(1)}(A,\Theta)=-[2\alpha\omega_0+f(A\cos\Phi,-A\omega_0\sin\Phi)]\sin\Phi\cos\Phi \tag{5b}$$

$$m_1^{(2)}(A,\Theta)=-\frac{u}{\omega_0}\sin\Phi \tag{5c}$$

$$m_2^{(2)}(A,\Theta)=-\frac{u}{A\omega_0}\cos\Phi \tag{5d}$$

$$\sigma_{1k}(A,\Theta)=-\frac{g_k(A\cos\Phi,-A\omega_0\sin\Phi)}{\omega_0}\sin\Phi \tag{5e}$$

$$\sigma_{2k}(A,\Theta)=-\frac{g_k(A\cos\Phi,-A\omega_0\sin\Phi)}{A\omega_0}\cos\Phi \tag{5f}$$

The right-hand sides of Eqs. (4a) and (4b) are indeed small under above assumptions, and both $A(t)$ and $\Theta(t)$ are slowing varying. According to [26], the vector process $[A,\Theta]$ will converge to a diffusive Markov process when $\varepsilon\to 0$. Carrying out the stochastic averaging method, the amplitude process $A(t)$ itself is a Markov process, governed by the following Itô stochastic equation:

$$dA=[m^{(1)}(A)+m^{(2)}(A)]dt+\sigma(A)dB(t) \tag{6}$$

where $B(t)$ is a unit Wiener process, and $m^{(1)}(A)$, $m^{(2)}(A)$ and $\sigma(A)$ are derived as,

$$m^{(1)}(A)=\langle m_1^{(1)}\rangle_t+\int_{-\infty}^{0}\sum_{r,k=1}^{n}\left\langle\frac{\partial\sigma_{1k}(t)}{\partial A}\sigma_{1r}(t+\tau)+\frac{\partial\sigma_{1k}(t)}{\partial\Theta}\sigma_{2r}(t+\tau)\right\rangle_t R_{kr}(\tau)d\tau \tag{7a}$$

$$m^{(2)}(A)=\langle m_1^{(2)}\rangle_t \tag{7b}$$

$$\sigma^2(A)=\int_{-\infty}^{-\infty}\sum_{r,k=1}^{n}\langle\sigma_{1k}(t)\sigma_{1r}(t+\tau)\rangle_t R_{kr}(\tau)d\tau \tag{7c}$$

where $\langle\bullet\rangle_t$ denotes the time averaging in one period, defined as,

$$\langle\bullet\rangle_t=\frac{1}{2\pi}\int_0^{2\pi}\bullet d\Phi \tag{8}$$

The amplitude process $A(t)$ will be investigated hereafter. It is noted that the control force u is included in the term $m^{(2)}(A)$, and it will become a function of the amplitude A after the time averaging.

2.2. Optimal bounded control

Since the response amplitude $A(t)$ is an implication of the response magnitude level, reduction of $A(t)$ results in a lower system response, which is the purpose of the feedback control. In order to deal with the nonstationary responses of the system, a time interval should be set in prior. Since we are interested in any time instant T, the time interval should be $[t_0,T]$. Let us assume$t_0\le t_1\le\ldots\le t_i\le t_{i+1}\le\ldots\le T$, and consider the optimal control problem in any subinterval $[t_i,t_{i+1}]$ with the following form of performance index:

$$J=E\left\{\int_{t_i}^{t_{i+1}}F(a,u,t)dt+G[a(T)]\right\} \tag{9}$$

where a is the state variable of the random amplitude $A(t)$. F and G are two continuous differential convex functions. Based on the dynamical programming principle [27], together with Eq. (6), a simplified dynamical programming equation can be derived in terms of the optimal performance index η as follows,

$$\eta=J(u^*)=\min_u\left[F(a,u,t)+\overline{m}(a)\frac{dV}{da}+\frac{1}{2}\sigma^2(a)\frac{d^2V}{da^2}\right] \tag{10}$$

where, $\overline{m}(a)=[m^{(1)}(A)+m^{(2)}(A)]_{A=a}$, u^* is the optimal control force, and $V=V(a)$ is the value function. Assume that $u_0(u_0>0)$ is the allowed maximum control force so that $u(A)|\le u_0$. By minimizing the right hand side of Eq. (10) with respect to u, noticing only one term $m^{(2)}(a)$ associated with u, and imposing the constraint $u(A)|\le u_0$, the optimal control force u^*in $[t_i,t_{i+1}]$ can then be deduced as,

$$u^*=-u_0\mathrm{sgn}\left(\frac{\dot{x}}{\omega_0^2}\frac{dV}{da}\right) \tag{11}$$

where 'sgn' denotes the sign function. Since the value function $V(a)$ should increase with increasing amplitude, $dV/da>0$ [23,24],