Probabilistic Engineering Mechanics 38 (2014) 42–53

Contents lists available at ScienceDirect

Probabilistic Engineering Mechanics

journal homepage: www.elsevier.com/locate/probengmech

Modified linear estimation method for generating multi-dimensional multi-variate Gaussian field in modelling material properties

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ARTICLE INFO

Article history:
Received 1 August 2014
Accepted 2 September 2014
Available online 10 September 2014

Keywords:
Random field
Material property
Autocorrelation function
Random finite element analysis

ABSTRACT

Although a number of methods have been developed to generate random fields, it remains a challenge to efficiently generate a large, multi-dimensional, multi-variate property field. For such problems, the widely used spectral representation method tends to require relatively longer computing time. In this paper, a modified linear estimation method is proposed, which involves mapping the linearly estimated field through a series of randomized translations and rotations from one realization to the next. These randomized translations and rotations enable the simulated property field to be stationary. The autocorrelation function of the simulated fields can be approximately described by a squared exponential function. The algorithms of the proposed method in both the rectangular and cylindrical polar coordinate systems are demonstrated and the results validated by Monte-Carlo simulations. Comparisons between the proposed method and spectral representation method show that the results from both methods are in good agreement, as long as the cut-off wave numbers of the spectral representation method are sufficiently large. However, the proposed method requires much less computational time than the spectral representation method. This makes it potentially useful for generating large multi-dimensional fields in random finite element analysis. Applications of the proposed method are exemplified in both rectangular and cylindrical polar coordinate systems.

1. Introduction

Many materials possess significant spatial variability in properties [8,9,35,36]. An example in geotechnical engineering is cement-admixed soils, the strength of which can range from about 700 kPa to 5 MPa ([5]). These materials can be simulated by second-order stationary random fields [50] with a marginal distribution and an autocorrelation function (or correlation structure for multi-variate fields). Monte-Carlo simulations are usually required in random finite element analysis of such spatially variable media [37,13]. Thus, in conducting random finite element analysis of large three-dimensional problems, efficiency in generating random fields is a matter of practical importance.

A common method for generating second-order stationary random fields is the spectral representation method (SRM) [41–46,14,15]. The autocorrelation function of the generated field can be varied via the corresponding power spectral density function. Non-Gaussian random field can be derived from Gaussian random field by iteratively applying a memoryless translation [16,4]. Another commonly used approach is the Karhunen–Loeve (KL) expansion [12,34,33,47], which can generate both Gaussian and non-Gaussian random fields directly. As the scale of fluctuation (SOF) of a random field decreases, the KL expansion reduces to the SRM [49,19]. However, both methods involve summation of an infinite number of terms. This can lead to time-consuming computation for large three-dimensional random fields. Hence, they are seldom applied to such problems.

Matheron [31] proposed a turning bands method that requires the user to pre-define a set of basis lines using an existing method, such as the SRM. If the number of basis lines is not large enough, streaking may occur. Fenton and Vanmarcke [11] developed an efficient local average subdivision method, but this can generate systematic bias in variance. Liu et al. [26] proposed linear estimation method (LEM) which was subsequently improved by Li and Der Kiureghian [25]. This method involves a linear combination of several correlated random variables, and is thus relatively efficient to implement. However, it requires an initial grid of correlated random values. Lawrence [23,24] proposed a basis variable approach, by expanding a random function into a Fourier-type series, which is similar in concept to the KL expansion.

Most of those methods can generate a Gaussian random field with various autocorrelation functions. However, the autocorrelation function of a random property is often not readily determined from limited data [2]. Furthermore, Fenton and Griffith [10] showed that different types of autocorrelation function have insignificant

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http://dx.doi.org/10.1016/j.probengmech.2014.09.001
0266-8920/

effect on the overall performance of a material if the SOF is fixed. Hence, the focus is often on the SOF rather than autocorrelation function itself [22,1,48,30].

This paper proposes an efficient method for generating large three-dimensional random fields with controllable SOF. The proposed method, hereafter termed modified linear estimation (MLE) method, is able to generate a spatially continuous, stationary, ergodic and Gaussian random field, while preserving the efficiency of LEM. The basis of the method was the first presented, followed by adaptation from rectangular coordinate system to cylindrical polar coordinate system, which is needed to generate columnar-structured random fields of cement-admixed soil columns. The proposed method was then verified by using Monte-Carlo simulations and compared with the SRM. Finally, the efficiency of the proposed method in generating large three-dimensional random fields is demonstrated by two practical examples.

2. Linear estimation method

In the original LEM ([26]), a random variable f is calculated via the relation

$$f(\mathbf{x}) = \sum_{i=1}^{n} N_i(\mathbf{x}) \cdot f_{i,k} \tag{1}$$

where $\mathbf{x}$ is the position vector of a point within an n-noded finite element k, $f_{i,k}$ is the value of the random variable at the ith nodal point of element k, and $N_i(\mathbf{x})$ are the shape functions for element k [53]. The shape functions satisfy the condition

$$\sum_{i=1}^{n} N_i(\mathbf{x}) = 1 \tag{2}$$

The mean and variance of $f(\mathbf{x})$ in Eq. (1) can be written as

$$\bar{E}[f(\mathbf{x})] = \sum_{i=1}^{n} N_i(\mathbf{x}) \cdot \bar{E}[f_{i,k}] \tag{3}$$

$$D[f(\mathbf{x})] = \sigma^2 \left(\sum_{i=1}^{n} N_i^2(\mathbf{x}) + \sum_{j>i}^{n} \sum_{i=1}^{n-1} 2\rho_{ij} N_i(\mathbf{x}) N_j(\mathbf{x}) \right) \tag{4}$$

where $\bar{E}[\ ]$ and $D[\ ]$ represent the expectation and variance operators, respectively; σ is the standard deviation of the variate at every node ($f_{i,k}$ are assumed to have constant standard deviation); $\rho_{ij}=\rho_{ji}$ is the correlation coefficient between the variables at the ith and jth nodal points.

Equation (3) shows that in the LEM, a constant mean is implied by virtue of Eq. (2); however, Eq. (4) implies the variance of the simulated random field $f(\mathrm{x})$ differs from σ^2 and also is not constant unless $\rho_{ij}=1$, which is not always the case. As a result, $f(\mathrm{x})$ is not always stationary. Li and Der Kiureghian [25] recommended using optimized coefficients instead of shape functions in Eq. (1) so that the error in variance can be minimized. In the original LEM as well as the modification of Li and Der Kiureghian [25], the correlated nodal values need to be generated, such as by SRM.

3. Modified linear estimation method

The basic principle of the proposed MLE method is to generate a stationary Gaussian random field with zero-mean, unit-variance and SOF $=\sqrt{\pi}$. This type of random field will be hereafter termed *property field* with position vector $\mathbf{y}$. Random fields with any prescribed SOF can be readily modeled by stretching from this basic *property field*.

3.1. Monte-Carlo algorithm for n-dimensional m-variate Gaussian random field

To generate an n-dimensional m-variate (nDmV) *property field* with a cross-correlation matrix $\mathbf{C}$ in the MLE method involves the following four steps.

Step 1. Discretize the n-dimensional hyperspace with position vector $\mathbf{s}$ into an n-dimensional grid with unit grid spacing and populate each grid node with an m-component $\underline{N}(0,1)$ vector $\mathbf{r}$, where $\underline{N}(0,1)$ denotes a random number generated from the Gaussian distribution with zero-mean and unit-variance; that is, the components in $\mathbf{r}$ are independent and identically distributed (i.i.d).

Step 2. Perform the Cholesky decomposition [7] on the cross-correlation matrix $\mathbf{C}$, giving

$$\mathbf{C} = \mathbf{L} \cdot \mathbf{L}^T \tag{5}$$

and replace the random vector $\mathbf{r}$ in each node by $\mathbf{f}$, such that

$$\mathbf{f} = \mathbf{L} \cdot \mathbf{r} \tag{6}$$

where $\mathbf{L}$ is the lower triangular matrix. Physically this transforms $\mathbf{r}$ from being an uncorrelated random vector to a correlated random vector such that its correlation matrix is $\mathbf{C}$. This hyperspace together with its nodal vectors will be hereafter referred to as a *precursor random field* with position vector $\mathbf{s}$.

Step 3. Locate the position of *property field* ($\mathbf{y}$) in the *precursor random field* ($\mathbf{s}$) via the relation

$$\mathbf{s} = \mathbf{J} \cdot \mathbf{y} + \varepsilon \tag{7}$$

where $\mathbf{J}=1$ for one-dimensional field, and

$$\mathbf{J} = \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix} \tag{8}$$

and

$$\mathbf{J} = \begin{bmatrix} \cos(\psi_3) & -\sin(\psi_3) & 0 \\ \sin(\psi_3) & \cos(\psi_3) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\psi_2) & 0 & \sin(\psi_2) \\ 0 & 1 & 0 \\ -\sin(\psi_2) & 0 & \cos(\psi_2) \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\psi_1) & \sin(\psi_1) \\ 0 & -\sin(\psi_1) & \cos(\psi_1) \end{bmatrix} \tag{9}$$

for two- and three-dimensional fields, respectively. The components in $\mathbf{J}$ (namely, ψ, ψ_1, ψ_2 and ψ_3) represent the angles of rotation; $\boldsymbol{\varepsilon}$ is a translation vector with n components. In effect, for fixed values of $\mathbf{J}$ and $\boldsymbol{\varepsilon}$, the *property field* overlies the *precursor random field* with a certain rotation angle and translation in position as shown in Fig. 1. The components in $\mathbf{J}$ and $\boldsymbol{\varepsilon}$ can be determined in such a way that they are fixed in each realization, but change from one realization to the next. In other words, they are independent random variables, but they have specific values in each realization. In this study, components in $\mathbf{J}$ and $\boldsymbol{\varepsilon}$ follow the uniform distribution in the ranges $[0, \pi/2]$ and $[0, 1]$, respectively.

Step 4. Once the relative position between these two fields is fixed with Step 3, a continuous *property field* with an arbitrary coordinate $\mathbf{y}$ (Fig. 1) can be obtained via the relation

$$\mathbf{f}^j(\mathbf{y}) = \sum_{i=1}^{2^n} \sqrt{N_i(\mathbf{y})} \cdot \mathbf{f}_{i,k}^j, j = 1, 2, \ldots, m. \tag{10}$$

where $\mathbf{f}^j(\mathbf{y})$ and $\mathbf{f}_{i,k}^j$ are the jth components of the random vectors at point $\mathbf{y}$ and at node i of element k, respectively; $N_i(\mathbf{y})$ are the shape functions of a 2^n-noded element taken from the pool of well-established shape functions in finite element method (e.g. [53]). By applying Eq. (10), each realization of the *property field* is a continuous function of spatial coordinates, which is physically reasonable. The square root of the shape functions is introduced to ensure that $\mathbf{f}^j(\mathbf{y})$ has a unit variance.