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Fokker Planck equation solved in terms of complex fractional moments

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ABSTRACT

In this paper the solution of the Fokker Planck (FPK) equation in terms of (complex) fractional moments is presented. It is shown that by using concepts coming from fractional calculus, complex Mellin transform and related ones, the solution of the FPK equation in terms of a finite number of complex moments may be easily found. It is shown that the probability density function (PDF) solution of the FPK equation is restored in the whole domain, including the trend at infinity with the exception of the value of the PDF in zero.

1. Introduction

Stochastic differential calculus was born in the second part of the last century. The papers of Itô, Wong and Zakaj, Kolmogorov and many others opened the way to the one of main branches of interest in stochastic mechanics (see [1–4]). In the particular case of nonlinear systems einforced by external or parametric normal white noise the equation ruling the evolution of the conditional PDF is the FPK equation.

Solution of such equation or its generalizations to the case of Poisson white noise (Kolmogorov–Feller equation) or to α-stable white noise (usually termed as Einstein–Smoluchowsky equation) in terms of moments may not be pursued since in the case of normal white noise the resulting equations are hierarchical in the sense that the equation of moments of fixed order contains moments of higher order and then some truncation procedure has to be enforced. Other strategies such as cumulant neglect closure or Hermite polynomials may solve only particular cases and in any case the probability density function may exhibit negative values. At least usually, with the classical methods of solution, the trend for large value of the domain of PDF, is not guaranteed [5–9]. The latter aspect destroy the possibility to perform reliability analysis that is the central point of the structural analysis. Even though other methods are available in literature for the solution of the Fokker Planck equation like finite element method [10], stochastic averaging method [11–13], path integral solution [13–16], Wiener path integral technique [17], the FPK equation is difficult to solve in easy and direct way.

It is well known, that by knowing the moments of a random variable, the PDF may not be reconstructed. Some improvements for the PDF characterization may be obtained by using fractional moments of real order [18,19].

Recently it has been shown that by using concepts coming from fractional calculus in complex domain and Mellin transform a new form of expansion of the probability density function is obtained involving moments of the type $E[|X|^{\gamma-1}]$, $\gamma=\rho+i\eta$ termed as *Complex Fractional Moments* (CFM) [20–22]. These moments are complex quantities and are related to fractional Riesz integrals in zero and to the Mellin transform of the PDF. It has been also shown that with the CFM, evaluated for different value of the imaginary part (while $Re(\gamma)=\rho$ remains fixed), both PDF and characteristic function may be reconstructed in the whole respective domains by using the inverse Mellin transform [23]. Using similar concepts Correlation and Power Spectral Density function may be also represented as a summation of finite number of (complex) power law terms [24]. Then working in complex plane new informations in probability and their Fourier transform comes out also for heavy tails distribution (Lévy random variables).

Up to now these concepts seem to be related only to the description of the PDF, in this paper it will be shown that by working with CFM the solution of the FPK equation may be obtained by solving a finite set of ordinary differential equations involving only a limited number of moments of the type $E[|X|^{\gamma_k-1}]$, with $\gamma_k=\rho+i\eta_k$. This goal is obtained by making the Mellin transform of the FPK equation. Such a way has not been used in the past since the Mellin transform of the various terms of the FPK equation are evaluated for different values of the real part of the Mellin transform. This problem also happens for fractional or

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ordinary differential equations in the Mellin domain. It follows that only very particular cases of differential equations (see [25]) may be solved. Here this problem is overcome showing that CFM evaluated for a fixed value of ρ may reconstruct CFM for any other value of ρ as simple linear combinations.

2. Mellin transform and related concepts

Let $f(x)$ be any real function defined in $0 \le x < \infty$. The Mellin transform, labeled as $M_f(\gamma - 1)$, is defined as

$$\mathcal{M}\{f(x);\gamma\} = M_f(\gamma - 1) = \int_0^\infty f(x)\, x^{\gamma-1}\, dx; \quad \gamma = \rho + i\eta \tag{1}$$

where $i = \sqrt{-1}$ and $\rho, \eta \in \mathbb{R}$.

If the Mellin transform exists, then $f(x)$ may be obtained in the form

$$f(x) = \mathcal{M}^{-1}\left\{M_f(\gamma - 1); x\right\} = \frac{1}{2\pi}\int_{\eta=-\infty}^{\infty} M_f(\gamma - 1)x^{-\gamma}\, d\eta; \quad x > 0 \tag{2}$$

It is to be emphasized that the integration is performed along the imaginary axis while ρ remains fixed. The condition for the existence of the Mellin transform is that $-p < \rho < -q$, where p and q are the order of zero at $x=0$ and $x=\infty$, respectively. Namely

$$\lim_{x\to 0} f(x) = O(x^p); \quad \lim_{x\to\infty} f(x) = O(x^q) \tag{3}$$

where $O(\cdot)$ means order of the term in parenthesis.

Such an example if $f(x) = (1 + x)^{-1}$, since $\lim_{x\to 0} f(x) = 1[O(x^0)]$ then $p = 0$, and $\lim_{x\to\infty} f(x) = x^{-1}[O(x^{-1})]$, then $q = -1$; it follows that in this case the existence condition is $0 < \rho < 1$. The strip of ρ such that $-p < \rho < -q$ is the so called *Fundamental Strip* (FS) of the Mellin transform. If $-q$ is lesser than $-p$ the Mellin transform and its inverse do not exist.

Eq. (2) may be discretized in the form

$$f(x) \cong \frac{\Delta\eta}{2\pi}\sum_{k=-m}^{m} M_f(\gamma_k - 1)x^{-\gamma_k}; \quad \gamma_k = \rho + ik\Delta\eta \tag{4}$$

where $\Delta\eta$ is the discretization step along to the imaginary axis, $m\Delta\eta = \bar{\eta}$ is a cut-off value chosen in such a way that the contribution of terms of higher order than m do not produce sensible variations on $f(x)$. It is to be remarked that $M_f(\gamma - 1)$ is analytic onto the fundamental strip, and is such that

$$M_f(\rho + i\eta - 1) = M_f^*(\rho - i\eta - 1) \tag{5}$$

where the star means complex conjugate. It follows that with simple manipulations the summation in Eq. (4) may be rewritten in a summation from 0 to m.

The Riesz fractional integral of a certain function $f(x)$ that is zero for $x < 0$, denoted as $(I^\gamma f)(x)$, is defined as

$$(I^\gamma f)(x) = \frac{1}{2\nu_c(\gamma)}\int_0^\infty f(\xi)|x - \xi|^{\gamma-1}\, d\xi; \quad \rho > 0, \quad \rho \ne 1,\ 3, \ldots \tag{6}$$

where $\nu_c(\gamma) = \Gamma(\gamma)\cos(\gamma(\pi/2))$ and $\Gamma(\cdot)$ is the Euler Gamma function. By comparing Eqs. (1) and (6) it may be stated that the Mellin transform is related to Riesz fractional integral in zero, that is

$$2\nu_c(\gamma)(I^\gamma f)(0) = M_f(\gamma - 1) \tag{7}$$

Under this perspective the representation in Eq. (4) looks like a Taylor expansion because it involves an operator in zero and a (complex) power series on x; for more details see [23]. The main difference is that when a truncation on the classical Taylor series is performed, always the Taylor series diverges as x diverges, while no divergence problem occur using Eq. (4) since summation is performed along to the imaginary axis and ρ remain fixed. Moreover, unless $f(x)$ belongs to the class C_∞ in zero, the various derivatives in zero may be divergent quantities and the Taylor expansion in such cases is meaningless. On the contrary the series expressed in Eq. (4) never diverges provided ρ belongs to the FS of the Mellin transform and then $f(x)$ is reproduced in the whole domain with the exception of the value in zero. With these simple information we can now solve the FPK equation by using Mellin transform theorem.

3. Probability and complex fractional moments and its use for the solution of the Fokker Planck equation

In the ensuing derivations, for simplicity sake's, we suppose that the PDF of a stochastic process $X(t)$, in the following denoted as $p_X(x,t)$, is symmetric, namely $p_X(x,t) = p_X(-x,t)$.

The Mellin transform of $p_X(x,t)$, denoted as $M_{p_X}(\gamma - 1)$, is given in the form

$$M_{p_X}(\gamma - 1, t) = \int_0^\infty p_X(x,t)x^{\gamma-1}\, dx = \frac{1}{2}E[|X(t)|^{\gamma-1}] \tag{8}$$

where $E[\cdot]$ means ensemble average. From this equation it may be stated that the Mellin transform of the PDF is strictly related to moments of the type $E[|X(t)|^{\gamma-1}]$.

According to Eq. (4) the discretized version of the inverse Mellin Transform is written for $x > 0$ in the equivalent forms

$$p_X(x,t) = \frac{1}{4b}\sum_{k=-m}^{m} E[|X(t)|^{\gamma_k - 1}]x^{-\gamma_k} = \frac{1}{2b}x^{-\rho}\sum_{k=-m}^{m} M_{p_X}(\gamma_k - 1, t)x^{-i(k\pi/b)};$$
$$\gamma_k = \rho + i\frac{k\pi}{b} \tag{9}$$

where $b = \pi/\Delta\eta$ and ρ belongs to the FS of $p_X(x,t)$. Since $p_X(x,t) \ge 0$ and the area of the PDF in 0–∞ is 1/2 then $\lim_{x\to\infty} p_X(x,t) = 0$. It follows that the fundamental strip of $p_X(x,t)$ always exists and, for $p_X(0,t) \ne 0$, is $0 < \rho < u$. The value of u depends of the order of zero of the PDF at $x = \infty$. As an example for α-stable random variable the moments $E[|X|^\beta](\beta \in \mathfrak{R})$ do not diverge only in the range $-1 < \beta < \alpha$[23]. Then for such random variable the FS is $0 < \rho < \alpha + 1$. In general if for a given stochastic process the integer moments diverge starting from a certain value, say r, then the strictest FS is $0 < \rho < r + 1$.

Let us now suppose that the equation of motion of a (mass-less) non-linear half oscillator is given in the form

$$\begin{cases} \dot{X} = f(X,t) + W(t) \\ X(0) = X_0 \end{cases} \tag{10a,b}$$

where $W(t)$ is a normal zero mean white noise, formal derivative of the Brownian motion $B(t)$, $(dB(t)/dt = W(t))$ characterized by $E[dB^2(t)] = D\, dt$, being D the intensity of the white noise. In Eq. (10) we suppose that $f(X,t) = -f(-X,t)$ is a deterministic non-linear function of the stochastic output process $X(t)$. X_0 is a random variable with assigned distribution $(p_X(x,0) = p_X(-x,0))$. Under these assumptions the output stochastic process has a symmetric distribution $p_X(x,t)$.

The Fokker–Planck equation, ruling the transition probability of $X(t)$, is written in the form

$$\begin{cases} \dfrac{\partial p_X(x,t)}{\partial t} = -\dfrac{\partial}{\partial x}(f(x,t)p_X(x,t)) + \dfrac{D}{2}\dfrac{\partial^2 p_X(x,t)}{\partial x^2} \\ p_X(x,0) = \overline{p_X}(x) \end{cases} \tag{11a,b}$$

where the overbar means *assigned* PDF in $t = 0$.