Probabilistic Engineering Mechanics 38 (2014) 119–126

Contents lists available at ScienceDirect

Probabilistic Engineering Mechanics

journal homepage: www.elsevier.com/locate/probengmech

Poisson white noise parametric input and response by using complex fractional moments

Alberto Di Matteo*, Mario Di Paola [1], Antonina Pirrotta [2]

Dipartimento di Ingegneria Civile, Ambientale, Aerospaziale, deiMateriali (DICAM), Università degli Studi di Palermo, Viale delle Scienze, 90128 Palermo, Italy

ARTICLE INFO

Article history:
Received 16 January 2014
Received in revised form
16 June 2014
Accepted 17 July 2014
Available online 24 July 2014

Keywords:
Complex fractional moment
Mellin transform
Parametric Poisson white noise
Probability density function
Modified Kolmogorov–Feller equation

ABSTRACT

In this paper the solution of the generalization of the Kolmogorov–Feller equation to the case of parametric input is treated. The solution is obtained by using complex Mellin transform and complex fractional moments. Applying an invertible nonlinear transformation, it is possible to convert the original system into an artificial one driven by an external Poisson white noise process. Then, the problem of finding the evolution of the probability density function (PDF) for nonlinear systems driven by parametric non-normal white noise process may be addressed in determining the PDF evolution of a corresponding artificial system with external type of loading.

1. Introduction

In many cases of engineering interest the exciting forces depend upon the configuration of the structure, such as the case of follower forces. These systems are usually referred as parametric or multiplicative ones. A common procedure, in case of systems driven by a normal parametric white noise, consists in modifying the drift term of the stochastic differential equation (SDE) by adding the Wong–Zakai or Stratonovich (WZ–S) corrective term which accounts for the irregularities of the Brownian motion [1]. Once the original SDE is modified, the nonanticipating fundamental property of the Itô stochastic differential calculus (SDC) may be used, the Fokker–Plank equation is then readily found for the modified system, and the response statistics may be pursued in a very easy way. In the case of external Poissonian white noise input (additive case), the equation ruling the evolution of the probability density function (PDF) is the so called Kolmogorov–Feller (K–F) equation, in which the counterpart of the diffusive term is a convolution integral whose kernel is the probability of the spike occurrences. For these systems exact solutions may be found in literature for a very restricted class of nonlinearities and of PDF of impulse amplitude [2,3], whereas some numerical methods have been developed [4–9].

In the case of Poissonian white noise parametric input a hierarchy of corrective terms is necessary to get the jumps for each Dirac's delta occurrence [10–15]. Once the drift term is modified, the Kolmogorov–Feller equation may be obtained taking into account the impulsive character of the input. It is worth noting that the great majority of numerical methods available in literature deal with Gaussian or Poisson white noise external excitation, while the relevant analysis on multiplicative excitation, especially for the Poisson white noise case, is less addressed. Reader could refer to [16], and references therein, for a report on the numerical procedures available.

This paper aims at solving the aforementioned modified K–F equation with the aid of the so-called complex fractional moments (CFMs) [17,18]. These complex quantities are nothing else than moments of the type $E[|X|^{\gamma}]$, $\gamma \in \mathbb{C}$, where $X \in \mathbb{R}$ is a real random variable whose probabilistic characterization may be given both by the PDF $p_X(x)$ and its Fourier transform, namely the characteristic function (CF) $\phi_X(\vartheta)$. In particular in [17] it has been shown that CFMs are directly related to the Riesz fractional integral (see Appendix A) of order γ of the CF in zero, that is $(I^{\gamma}\phi_X)(0) = -E[|X|^{-\gamma}]$, $\mathrm{Re}\gamma > 0$. Further in [18] relations between CFMs and the Mellin transform of the PDF has been demonstrated in the form $E[|X|^{-\gamma}] = 2\mathscr{M}_u(-\gamma+1)$, where $\mathscr{M}_u(-\gamma+1) = \int_0^{\infty} u(x)x^{-\gamma}\,dx$ is the Mellin transform of the even part $u(x)$ of the PDF $p_X(x)$.

The appealing in working with these quantities instead of the classical moments are two-fold: (i) the CFMs never diverge (also for α-stable processes), provided the real part of γ belongs to the so-called fundamental strip of the Mellin transform; and (ii) CFMs are able to represent both PDF and the CF.

* Corresponding author.
E-mail addresses: alberto.dimatteo@unipa.it (A. Di Matteo), mario.dipaola@unipa.it (M. Di Paola), antonina.pirrotta@unipa.it (A. Pirrotta).
[1] Tel.: +39 091 23896737.
[2] Tel.: +39 091 23896756.

http://dx.doi.org/10.1016/j.probengmech.2014.07.003
0266-8920/

It is worth noting that recently the authors [19] introduced this approach for the solution of the K–F equation for nonlinear systems driven by external Poisson white noise process. Further, as well described in [20,21], it is possible to relate systems driven by external white noise to nonlinear systems driven by parametric type of excitation, through a nonlinear invertible transformation. In this way, once the PDF of the corresponding artificial system with external excitation is found, the PDF of the nonlinear system driven by parametric white noise may be readily obtained.

In this paper, taking full advantages of the method developed in [19] together with the nonlinear invertible transformation procedure described in [20,21], the evolution of the response PDF of nonlinear systems under parametric Poissonian white noise is restituted.

In order to assess the validity of the proposed method application to a nonlinear system driven by parametric Poisson white noise is presented and solution in terms of PDF is compared with that obtained with pertinent Monte Carlo simulations.

2. Kolmogorov–Feller equation

In this section the K–F equation and its generalization to the case of parametric input is briefly introduced for sake of completeness.

2.1. Kolmogorov–Feller equation (external excitation)

Let $W(t)$ be a Poisson white noise process. It is constituted by a train of impulses of random amplitude Y, with assigned PDF $p_Y(y,t)$. The impulse occurrence is distributed in time according to a Poisson law. Then each impulse Y_k occurs at a time T_k. The two random variables Y and T are independent each another. Under these assumptions the Poisson white noise is given as

$$W(t)=\sum_{k=1}^{N(t)} Y_k\,\delta(t-T_k) \tag{1}$$

where $\delta(\bullet)$ is the Dirac's delta and $N(t)$ is a Poisson counting process giving the number of impulses in $0/t$. The Poisson white noise may be considered as the formal derivative of the Compound Poisson process $C(t)$. Then

$$C(t)=\sum_{k=1}^{N(t)} Y_k\,U(t-T_k) \tag{2}$$

where $U(\bullet)$ is the unit step function. Increment of the Compound Poisson process are characterized by

$$E[dC(t)^j]=\lambda(t)E[Y(t)^j]dt \tag{3}$$

where $E[\bullet]$ stochastic average and $\lambda(t)$ the mean number of impulses per unit time. If λ is a constant, then the Poisson white noise is stationary.

Let the equation of motion of a nonlinear system driven by the Poisson white noise be given in the form

$$\begin{cases}\dot{X}=f(X,t)+W(t)\\ X(0)=X_0\end{cases} \tag{4}$$

where $f(X,t)$ is a nonlinear function of the process $X(t)$ and X_0 is the initial condition, that is a random variable with assigned PDF in zero $p_X(x,0)=\overline{p}(x)$.

The equation ruling the evolution of the PDF of the response process $X(t)$ is the so-called Kolmogorov–Feller equation, that may be written as

$$\begin{cases}\frac{\partial p_X(x,t)}{\partial t}=-\frac{\partial}{\partial x}(f(x,t)p_X(x,t))-\lambda(t)p_X(x,t)+\lambda(t)\int_{-\infty}^{\infty}p_Y(\xi)p_X(x-\xi,t)d\xi\\ p_X(x,0)=\overline{p}(x)\end{cases} \tag{5}$$

For simplicity sake's we suppose that $\overline{p}(x)$ and $p_Y(y)$ have symmetric distributions and $f(x,t)=-f(-x,t)$ $\forall t$. Under these assumptions $p_X(x,t)=p_X(-x,t)$ $\forall t$. The case of non symmetric distribution may be faced by considering the paper [18].

2.2. Modified Kolmogorov–Feller equation (parametric excitation)

Consider the equation of motion of a nonlinear system driven by a parametric Poisson white noise process, that is

$$\begin{cases}\dot{X}=f(X,t)+g(X,t)\,W(t)\\ X(0)=X_0\end{cases} \tag{6}$$

In this case the impulses are modulated by a nonlinear function of the response $g(X,t)$. Let us suppose that $g(X,t)$ is ∞ time differentiable on X. If $g(X,t)=1$ $\forall t$ then the excitation is external, Eq. (5) remains valid and at each Dirac's delta occurrence at time T_k the sample function of $X(t)$ exhibits a jump that is exactly Y_k (amplitude of the spike at time T_k). If $g(X,t)$ is a function of $X(t)$ then the jump depends both on the value of $g(X,t)$ immediately before the impulse and on the amplitude of the spike [14]; in this case each jump at time T_k is given as

$$\Delta X_k=\sum_{j=1}^{\infty} Y_k^j\frac{g^{(j)}(X(T_k^-),T_k)}{j!} \tag{7}$$

where $g^{(j)}$ can be evaluated in recursive form as follows:

$$g^{(j)}(X(t),t)=\frac{\partial g^{(j-1)}(X(t),t)}{\partial X}g^{(1)}(X(t),t) \tag{8.a}$$

$$g^{(1)}(X(t),t)=g(X(t),t) \tag{8.b}$$

where $\Delta X_k=X(T_k^+)-X(T_k^-)$, the superscripts $+$ and $-$ stand for immediately after and before the jump occurrence, respectively.

As soon as the jump is predicted by knowing the intensity Y_k of the spike occurring at time T_k and the value of the response $X(T_k^-)$, that is the response immediately before the Dirac's delta occurrence, the non-anticipating property of the Itô calculus may be inferred. The Kolmogorov–Feller equation extended to the case of parametric Poisson white noise may be written in the form [21]

$$\frac{\partial p_X(x,t)}{\partial t}=-\frac{\partial}{\partial x}(f(x,t)p_X(x,t))+\lambda\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\frac{\partial^k}{\partial x^k}\left\{p_X(x,t)\left[\underbrace{\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\dots\sum_{p=1}^{\infty}}_{k-fold}\frac{g^{(j)}(x)g^{(l)}(x)\dots g^{(p)}(x)}{j!l!\cdots p!}\right]E[Y^{j+l+\dots+p}]\right\} \tag{9}$$

With this information we can now proceed towards the solution of this equation by using complex fractional moments. This issue will be addressed in the next section.

3. Mellin transform and complex fractional moment

Let $p_X(x,t)=p_X(-x,t)$, that is the response of Eq.(4) is symmetrically distributed. This may happen with some restrictions: both $f(X,t)$ and $g(X,t)$ are antisymmetric, the PDF of the impulses is such that $p_Y(y,t)=p_Y(-y,t)$ and the distribution of X_0 is symmetric. With these restrictions the approach in terms of complex fractional moments is quite simple. The case of non-symmetric distribution may be treated with the results given in [18].

If $p_X(x,t)$ is symmetric, we may evaluate $p_X(x,t)$ in the positive range $0\le x\le\infty$ by using the Mellin transform defined as

$$\mathscr{M}\{p_X(x,t);\ \gamma\}=\int_0^{\infty}p_X(x,t)x^{\gamma-1}\,dx=\mathscr{M}_{p_X}(\gamma-1,t) \tag{10}$$

where $\gamma=\rho+i\eta$ with $\rho,\ \eta\in\mathbb{R}$ and ρ belongs to the Fundamental Strip (FS) of the Mellin transform. In particular the FS is