Probabilistic Engineering Mechanics 38 (2014) 127–135

Contents lists available at ScienceDirect

Probabilistic Engineering Mechanics

journal homepage: www.elsevier.com/locate/probengmech

Stochastic response determination of nonlinear oscillators with fractional derivatives elements via the Wiener path integral

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ARTICLE INFO

Article history:
Received 7 December 2013
Received in revised form
25 June 2014
Accepted 1 July 2014
Available online 11 July 2014

Keywords:
Fractional derivative
Fractional variational problem
Euler–Lagrange equation
Wiener path integral
Stochastic dynamics
Nonlinear system

ABSTRACT

A novel approximate analytical technique for determining the non-stationary response probability density function (PDF) of randomly excited linear and nonlinear oscillators endowed with fractional derivatives elements is developed. Specifically, the concept of the Wiener path integral in conjunction with a variational formulation is utilized to derive an approximate closed form solution for the system response non-stationary PDF. Notably, the determination of the non-stationary response PDF is accomplished without the need to advance the solution in short time steps as it is required by the existing alternative numerical path integral solution schemes which rely on a discrete version of the Chapman–Kolmogorov (C–K) equation. This is accomplished by circumventing the solution of the associated Euler–Lagrange equation ordinarily used in the path integral based procedures. The accuracy of the technique is demonstrated by pertinent Monte Carlo simulations.

1. Introduction

Since the pioneering work by Gemant [1] and Bosworth [2], who first proposed fractional derivatives modeling for the constitutive behavior of viscoelastic media based on past results by Nutting [3], fractional calculus has been successfully applied in diverse fields such as viscoelasticity and rheology, control theory, biophysics, bioengineering, image and signal processing, and random walk models. A rather detailed account of various recent theoretical advances and applications of fractional calculus in the various fields can be found in the books by Sabatier et al. [4] and by Hilfer [5]. In this regard, applications of fractional derivatives in structural engineering for vibration control or seismic isolation purposes include modeling of the restoring force of structural systems equipped with viscoelastic dampers (e.g. [6–10]). In this regard, the theoretical modeling and analytical derivations have been found in quite good agreement with experimental data (e.g. Makris and Constantinou [11,12]).

Note that limitations pertaining to available information and the interpretation of prevalent mechanisms, as well as inherent uncertainty in critical engineering problems have necessitated the study of systems with stochastic parameters, input, and initial/boundary conditions. In this context, a stochastic approach constitutes a rational basis for system analysis and sustainable design. Nevertheless, complex nonlinear and hysteretic behavior observed in many systems renders such a stochastic analysis a persistent challenge. In this regard, it is pointed out that although theoretical research in the field of stochastic dynamics has already led to seminal advancements (e.g. [13,14]), the adoption and generalization of potent mathematical tools and concepts from theoretical physics, such as the Wiener/Feynman path integral [15–17], can offer a novel perspective and tools for engineering systems.

Monte Carlo simulation (MCS) techniques (e.g. Rubinstein and Kroese [18]) have been among the most versatile tools for determining, the response statistics of arbitrary stochastic systems. However, there are cases, especially for large scale complex systems, where MCS techniques can be computationally prohibitive. Thus, there is a need for developing alternative efficient approximate analytical and/or numerical solution techniques (e.g. see [19–21] for some recent references). In this regard, one of the promising frameworks relates to the concept of the Wiener path integral (WPI). It is noted that although the WPI has been well established in the field of theoretical

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http://dx.doi.org/10.1016/j.probengmech.2014.07.001
0266-8920/

physics, the engineering community has ignored its potential as a powerful uncertainty quantification tool. The concept of path integral was introduced by Wiener [15,16] and was reinvented in a different form by Feynman [17] to reformulate quantum mechanics. A detailed treatment of path integrals, especially of the Feynman path integral and its applications in physics, can be found in a number of books such as the one by Chaichian and Demichev [22]. Recently, an approximate analytical WPI technique for addressing certain stochastic engineering dynamics problems was developed by Kougioumtzoglou and Spanos [23]. The technique is based on a variational principle formulation in conjunction with a stochastic averaging/linearization treatment of the nonlinear equation of motion. In this regard, relying on the concept of the most probable trajectory an approximate expression was derived for the non-stationary response probability density function (PDF). Further, the aforementioned technique was extended by Kougioumtzoglou and Spanos [24] to treat multi-degree-of-freedom (MDOF) systems and hysteretic nonlinearities. The enhanced technique circumvents approximations associated with the stochastic averaging/linearization treatment of the previous development.

In passing it is noted that the aforementioned WPI technique should not be confused with alternative numerical schemes (commonly referred to as numerical path integral schemes) which constitute, in essence, a discrete version of the Chapman–Kolmogorov (C–K) Eqs. [25–29]. In this regard, utilizing the C–K equation the basic characteristic of those schemes is that the evolution of the PDF is computed in short time steps; thus, rendering the schemes computationally demanding potentially.

In this paper the WPI technique is further generalized to treat linear and nonlinear systems endowed with fractional derivatives terms subject to stochastic excitation. In this regard, it is noted that alternative existing approaches for determining the stochastic response of linear and nonlinear oscillators endowed with fractional derivatives elements resort either to stochastic averaging [30,31] or to statistical linearization [32] or to a simplification of the original single-degree-of-freedom-system (SDOF) by an increase of the system dimension [33,34]. These techniques exhibit various degrees of approximation or limitation. Thus, the herein developed WPI technique may offer a desirable alternative for determining the non-stationary response PDF of linear and nonlinear oscillators efficiently with a satisfactory degree of accuracy.

2. Analytical Wiener path integral formulation

2.1. Probability density functional

In the following the analytical WPI based technique, developed in Kougioumtzoglou and Spanos [20,24] is extended and generalized to account for linear and nonlinear SDOF systems endowed with fractional derivatives elements. In this regard, consider the nonlinear oscillator whose motion is governed by the differential equation

$$\ddot{x}(t)+C_\alpha({}^C_{ti}D^\alpha_t x)(t)+\omega_0^2 x(t)+f(x(t),{}^C_{ti}D^\alpha_t x(t))=w(t), \tag{1}$$

where a dot over a variable denotes differentiation with respect to time (t); $({}^C_{ti}D^\alpha_t x)$ is a restoring force governed by an α-order left Caputo fractional derivative defined as [35]

$${}^C_{ti}D^\alpha_t x(t)=\frac{1}{\Gamma(n-\alpha)}\int_{t_i}^{t}(t-\tau)^{n-\alpha-1}\frac{d^n}{d\tau^n}x(\tau)\,d\tau, \quad n-1<\alpha<n, \tag{2}$$

ω_0 is the natural frequency; C_α is a constant which can be viewed as a damping coefficient if $\alpha=1$, or as a stiffness coefficient if $\alpha=0$; $f(x,{}^C_{ti}D^\alpha_t x)$ represents a non-linear function depending on the instantaneous values of x and $({}^C_{ti}D^\alpha_t x)$; and $w(t)$ is a zero mean Gaussian white noise process of power spectral density S_0. Note that Eq. (1) reduces to the equation of motion of a conventional (non-fractional) nonlinear oscillator [24] when α approaches one.

Further, regarding the WPI [22], it can be realized as a functional integral over the space of all possible paths $C\{a_i,t_i;a_f,t_f\}$ starting at point $a(t_i)=a_i$ and having the endpoint $a(t_f)=a_f$, where $a(t)$ denotes an arbitrary stochastic process. It possesses a probability distribution on the path space as its integrand, which is denoted by $W[a(t)]$ and is called probability density functional. In this manner, the transition PDF is given by

$$p(a_f,t_f|a_i,t_i)=\int_{\{a_i,t_i\}}^{\{a_f,t_f\}}W[a(t)]\left[da(t)\right]. \tag{3}$$

Note that the probability density functional for the white noise process $w(t)$ is given by [22,36]

$$W[w(t)]=C\exp\left[-\int_{t_i}^{t_f}\frac{1}{2}\frac{w(t)^2}{2\pi S_0}dt\right], \tag{4}$$

where C is a normalization coefficient. Following next the approach proposed in Kougioumtzoglou and Spanos [24], Eq. (1) is substituted into Eq. (4) and the probability density functional $W[w(t)]$ for $w(t)$ is interpreted as the probability density functional $W[x(t)]$ for $x(t)$. This yields

$$W[x(t)]=C\exp\left[-\int_{t_i}^{t_f}\frac{1}{2}\frac{(\ddot{x}+C_\alpha({}^C_{ti}D^\alpha_t x)+\omega_0^2 x+f(x,{}^C_{ti}D^\alpha_t x))^2}{2\pi S_0}dt\right]. \tag{5}$$

2.2. Lagrangian formulation and fractional variational principle for the most probable path

It can be readily seen that even if the probability density functional is constructed, the analytical solution of the WPI of Eq. (3) is at least a rather daunting, if not impossible, procedure. Thus, to circumvent the aforementioned challenge, several research efforts have focused on developing approximate techniques for determining the transition PDF $p(a_f,t_f|a_i,t_i)$. In this regard, researchers invoked a variational formulation and defined a Lagrangian function for determining the most probable path, namely the most probable trajectory that connects the points $a(t_i)=a_i$ and $a(t_f)=a_f$. In this manner, a variational principle can lead to the associated Euler–Lagrange equation, whose solution is the most probable process realization; see Chachian and Demichev [22] and Kougioumtzoglou and Spanos [23] for a more detailed presentation.

Specifically, for the oscillator of Eq. (1), the corresponding Lagrangian function can be defined as

$$\mathscr{L}(x,{}^C_{ti}D^\alpha_t x,\ddot{x})=\frac{1}{2}\frac{(\ddot{x}+C_\alpha({}^C_{ti}D^\alpha_t x)+\omega_0^2 x+f(x,{}^C_{ti}D^\alpha_t x))^2}{2\pi S_0}. \tag{6}$$

Adopting next the variational formulation followed in Kougioumtzoglou and Spanos [20,24] the largest contribution to the Wiener path integral comes from the trajectory for which the integral in the exponential becomes as small as possible. Variational calculus rules [37] dictate that this trajectory with fixed end points satisfies the extremality condition

$$\delta\int_{t_i}^{t_f}\mathscr{L}(x_c,{}^C_{ti}D^\alpha_t x_c,\ddot{x}_c)\,dt=0, \tag{7}$$

where x_c denotes the most probable trajectory. In the ensuing analysis, the variational problem defined in Eq. (7) is coined fractional variational problem (FVP), since Eq. (7) contains an α-order left Caputo fractional derivative. This yields a corresponding Euler–Lagrange equation of the form

$$\frac{\partial\mathscr{L}}{\partial x_c}+{}_tD^\alpha_{t_f}\frac{\partial\mathscr{L}}{\partial\,{}^C_0D^\alpha_t x_c}+\frac{\partial^2}{\partial t^2}\frac{\partial\mathscr{L}}{\partial\ddot{x}_c}=0, \tag{8}$$