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A method for the evaluation of the response probability density function of some linear dynamic systems subjected to non-Gaussian random load

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ABSTRACT

This paper deals with the characterization of the random response of linear systems subjected to stochastic load. It proposes a new method based on the new version of the Probabilistic Transformation Method (PTM) that allows obtaining, with a very low computational effort, the probability density function of the response. An important aspect of the proposed approach is the ability to join directly the pdfs of the input load with those of the response. Based on the step-by-step integration method, explicit solutions will be proposed for the random response of systems loaded by seismic and windy sampled inputs.

1. Introduction

The characterization of the random response of a structural time-dependent system often requires a high computational effort. Actually, even for a system subjected to static actions the effort can be very high; this is related to the number of random variables involved and the type of probability distribution that characterizes them. The full probabilistic characterization of a random variable is given by the knowledge of its probability density function (pdf), or by its characteristic function (cf). Unfortunately there are no exact solutions, except for some simple cases, such as for linear systems subjected to Gaussian input.

The literature presents several methods that allow reconstructing the pdf response by using the moments (or cumulants) series method, with a relatively low computational effort [1–5]. The validity of these approaches was largely confirmed; however they lack a direct nature, namely the ability to join directly the pdf of the input with that of the output. Also, in the case of strong non-Gaussian response, a very high number of moments/cumulants are necessary and the convergence of these methods is not always guaranteed; and, at last, a very high computational effort is usually related to them. For the dynamic systems, sometimes, the evaluation of the random response is limited to the evaluation of the second order correlations and/or power spectral densities. In these cases, the literature shows several works, some of which providing exact solutions [6–8] or very powerful numerical procedures [9,10]. Monte Carlo methods [11,12] exhibit the well known problem that the accuracy of the estimates depends on the sampling size of the stochastic processes, besides the number of samples, increasing the related computational effort. Even these methods, moreover, do not define a direct input–output relationship in terms of pdf.

Aim of this work is to show the potential of the new version of the Probabilistic Transformation Method (PTM), first introduced by Falsone and Settineri for the static problem [13,14], to obtain the pdf response of some linear dynamic systems subjected to the non-Gaussian time-dependent input process.

2. Preliminary concepts

The differential equation governing the dynamic behavior of a multi-degrees-of-freedom linear system is usually written as follows:

$$\mathbf{M}\ddot{\mathbf{u}}(t)+\mathbf{C}\dot{\mathbf{u}}(t)+\mathbf{K}\mathbf{u}(t)=\mathbf{f}(t) \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the system mass, damping and stiffness $n\times n$ matrices, respectively, $\mathbf{u}(t)$ is the n-vector that collects the system degrees of freedom, and $\mathbf{f}(t)$ is the n-vector of the external loads. In the state space variables, the equation of motion is rewritten in the following form:

$$\dot{\mathbf{x}}(t)=\mathbf{D}\mathbf{x}(t)+\mathbf{v}\mathbf{f}(t) \tag{2}$$

where

$$\mathbf{x}(t)=\begin{pmatrix}\mathbf{u}(t)\\ \dot{\mathbf{u}}(t)\end{pmatrix};\mathbf{D}=\begin{pmatrix}\mathbf{0} & \mathbf{I}\\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C}\end{pmatrix};\mathbf{v}=\begin{pmatrix}\mathbf{0}\\ \mathbf{M}^{-1}\end{pmatrix} \tag{3a–c}$$

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The vector $\mathbf{x}(t)$, collecting the response state variables, can be evaluated by Duhamel's integral, that is

$$\mathbf{x}(t) = \mathbf{\Theta}(t)\mathbf{x}_0 + \int_0^t \mathbf{\Theta}(t-\tau)\mathbf{v}\mathbf{f}(\tau)d\tau \tag{4}$$

where $\mathbf{x}_0$ is the vector collecting the initial conditions at $t=0$, and $\mathbf{\Theta}(t)$ is the fundamental matrix related to the differential equation of motion, that can be defined in the following way:

$$\mathbf{\Theta}(t) = \exp(\mathbf{D}t) \tag{5}$$

The response system $\mathbf{x}(t)$ can be evaluated numerically by several methods; among these, the step-by-step integration method, based on the fundamental matrix, will be used in this work. This method enables us to solve in closed form, in a generic step Δt, the convolution integral given in Eq. (4), once a polynomial interpolation law is assumed for the vector load $\mathbf{f}(t)$ in correspondence of the same time step $\mathbf{\Delta}t$. As an example, assuming for $\mathbf{f}(t)$ a linear interpolation law within the interval $[t_{k-1}, t_k]$, one obtains the following step-by-step numerical procedure:

$$\mathbf{x}(t_k) = \mathbf{x}(k\Delta t) = \mathbf{\Theta}(\Delta t)\mathbf{x}(t_{k-1}) + \mathbf{\Gamma}_0(\Delta t)\mathbf{f}(t_{k-1}) + \mathbf{\Gamma}_1(\Delta t)\mathbf{f}(t_k) \tag{6}$$

$\mathbf{x}(t_k)$ being the system response at the time $t_k = k\Delta t$; analogously $\mathbf{f}(t_k)$ is the vector load evaluated at the same time. The vector operators $\mathbf{\Gamma}_0(\Delta t)$ and $\mathbf{\Gamma}_1(\Delta t)$ are given by the following relationships:

$$\mathbf{\Gamma}_1(\Delta t) = \left[\frac{\mathbf{L}(\Delta t)}{\Delta t} - \mathbf{I}\right]\mathbf{D}^{-1}\mathbf{v}; \mathbf{\Gamma}_0(\Delta t) = \left[\mathbf{\Theta}(\Delta t) - \frac{\mathbf{L}(\Delta t)}{\Delta t}\right]\mathbf{D}^{-1}\mathbf{v};$$
$$\mathbf{L}(\Delta t) = [\mathbf{\Theta}(\Delta t) - \mathbf{I}]\mathbf{D}^{-1} \tag{7a–c}$$

Appling recursively the step-by-step procedure given above, it is possible to define the relationship between the response system at the time t_k and all the vectors load $\mathbf{f}(t_i)$, $i = 0, 1, ..., k$, that is

$$\mathbf{x}(t_k) = \mathbf{x}(k\Delta t) = \mathbf{\Theta}^k(\Delta t)\mathbf{x}(t_0) + \mathbf{\Theta}^{k-1}(\Delta t)\mathbf{\Gamma}_0(\Delta t)\mathbf{f}(t_0) + \\ + \sum_{i=1}^{k-1}\left[\mathbf{\Theta}^{k-i}(\Delta t)\mathbf{\Gamma}_1(\Delta t) + \mathbf{\Theta}^{k-i-1}(\Delta t)\mathbf{\Gamma}_0(\Delta t)\right]\mathbf{f}(t_i) + \mathbf{\Gamma}_1(\Delta t)\mathbf{f}(t_k) \tag{8}$$

If the system is driven by a random vector load, even the structural response is a random process and it must be defined probabilistically. In this case, all the vectors load $\mathbf{f}(t_i)$, with $i = 0, 1, ..., k$, appearing in Eq. (8), can be considered as samples of the stochastic process $\mathbf{f}(t)$ extracted at the sampling step Δt; correspondently, the vector response $\mathbf{x}(t_k)$ represents a sample response of the stochastic process $\mathbf{x}(t)$. In order to characterize probabilistically the stochastic process $\mathbf{x}(t)$, one could characterize the sample response $\mathbf{x}(t_k)$; this is what made in this work, and, as will be shown later, the fundamental approach to obtain this result is the PTM, whose basic concepts are discussed in next section.

3. Basic concept of the PTM

The PTM is based on some relationships that enable to join the joint probability density functions (jpdfs) of two random vectors connected by a deterministic law. Let us consider an n-dimensional random vector $\mathbf{x}$ and a n-dimensional invertible application $\mathbf{g}^{-1}(\bullet) = \mathbf{h}(\bullet)$ such that

$$\mathbf{x} = \mathbf{h}(\mathbf{f}); \quad \mathbf{f} = \mathbf{g}(\mathbf{x}) \tag{9a – b}$$

$\mathbf{x}$ being a random vector, as well as $\mathbf{f}$. The jpdfs of $\mathbf{x}$ and $\mathbf{f}$, that are $p_{\mathbf{x}}(\mathbf{x})$ and $p_{\mathbf{f}}(\mathbf{f})$, are joined by the following relationships [15–20]:

$$p_{\mathbf{x}}(\mathbf{x}) = \frac{1}{|\det[\mathbf{J_h}(\mathbf{x})]|} p_{\mathbf{f}}(\mathbf{g}(\mathbf{x}))$$

$$p_{\mathbf{x}}(\mathbf{x}) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} ..(n).. \int_{-\infty}^{+\infty} p_{\mathbf{f}}(\mathbf{y})\delta(\mathbf{x} - \mathbf{h}(\mathbf{y}))d\mathbf{y} \tag{10a – b}$$

$\mathbf{J_h}(\mathbf{x})$ being the Jacobian matrix related to the transformations given in Eq. (9) and $\delta(\mathbf{x} - \mathbf{h}(\mathbf{y}))$ the n-dimensional Dirac Delta centered in the coordinate vector $\mathbf{h}(\mathbf{y})$, that are

$$\mathbf{J_h}(\mathbf{x}) = (\nabla_{\mathbf{f}}^T \otimes \mathbf{h}(\mathbf{f}))|_{\mathbf{f}=\mathbf{g}(\mathbf{x})};$$
$$\delta(\mathbf{x} - \mathbf{h}(\mathbf{y})) = \delta(x_1 - h_1(\mathbf{y}))\delta(x_2 - h_2(\mathbf{y}))..\delta(x_n - h_n(\mathbf{y})) \tag{11a – b}$$

In Eq. (11a), $\nabla_{\mathbf{f}}^T$ is the nth order row-vector differential operator collecting all the partial derivatives with respect to the component f_i of $\mathbf{f}$ and the symbol $\otimes$ indicates the Kronecker product; in Eq. (11b) $h_j(\mathbf{y})$ (with $j = 1, 2..., n$), is the jth element of the n-dimensional application $\mathbf{h}(\mathbf{y})$.

Eq. (10) provides a direct relation between the pdfs of the random vectors $\mathbf{x}$ and $\mathbf{f}$. Eq. (10a) requires that the random vectors have the same order; however this aspect is not a restriction and can be overcome easily [15]; it also requires that $\mathbf{h}(\bullet)$ has only one inverse application $\mathbf{g}(\bullet)$. If $\mathbf{h}(\bullet)$ has more than one inverse application, $p_{\mathbf{x}}(\mathbf{x})$ is defined as the summation of Eq. (10a) like relation over all the possible inversion points [15]. Eq. (10a) provides a direct relation between the jpdfs of the random vectors $\mathbf{x}$ and $\mathbf{f}$ by a multidimensional integral and does not require the knowledge of the Jacobian matrix. From Eq. (10b) it is possible to obtain the integral relationship of every marginal pdf by integrating respect to the all other variables and taking into account the properties of the Dirac Delta functions. For example, the first and second order marginal pdfs have the following integral relationships:

$$p_{x_j}(x_j) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} ..(n).. \int_{-\infty}^{+\infty} p_{\mathbf{f}}(\mathbf{y})\delta(x_j - h_j(\mathbf{y}))d\mathbf{y} \tag{12}$$

$$p_{x_jx_k}(x_j, x_k) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} ..(n).. \int_{-\infty}^{+\infty} p_{\mathbf{f}}(\mathbf{y})\delta(x_j - h_j(\mathbf{y}))\delta(x_k - h_k(\mathbf{y}))d\mathbf{y} \tag{13}$$

From Eqs. (12) and (13) it is possible to obtain the integral relationships of the first and second order characteristic functions; applying the Fourier transform to both sides of Eq. (12) and the double Fourier transform to both sides of Eq. (13), one obtains

$$M_{x_j}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} p_{x_j}(x_j)\exp(-i\omega x_j)dx_j \\ = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} ..(n).. \int_{-\infty}^{+\infty} p_{\mathbf{f}}(\mathbf{y})\exp(-i\omega h_j(\mathbf{y}))d\mathbf{y} \tag{14}$$

$$M_{x_jx_k}(\omega_j, \omega_k) = \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} p_{x_jx_k}(x_j, x_k)\exp(-i\omega_j x_j - i\omega_k x_k)dx_j dx_k \\ = \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} ..(n).. \int_{-\infty}^{+\infty} p_{\mathbf{f}}(\mathbf{y})\exp(-i\omega_j h_j(\mathbf{y}) \\ - i\omega_k h_k(\mathbf{y}))d\mathbf{y} \tag{15}$$

Eqs. (14) and (15) are the reference relations of the new version of the PTM proposed by Falsone and Settineri [19,20]. In the classical approach based on the PTM, the relationship (10a) is used, with the drawback related to the uniqueness of the inverse application, as discussed above. Moreover, to obtain any marginal pdf the evaluation of a multi-dimensional integral is necessary, and this can be computationally very heavy, above all of very large systems. Nevertheless, even Eqs. (12) and (13) require the solution of a multi-dimensional integral; however, in some cases Eqs. (14) and (15) can be solved in closed form, requiring only an inverse Fourier transform to obtain the marginal pdf.

For example, these relations give very interesting results in the case of linear transformations, that is if $\mathbf{f}$ and x are connected by a