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On the application of the probability transformation method for the analysis of discretized structures with uncertain proprieties

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ABSTRACT

The aim of this work is to show a novel approach for the analysis of random systems. This approach, based on the application of the Probabilistic Transformation Method (PTM), is here developed for the study of uncertain structural systems. These systems are characterized by the fact that some of their geometrical and/or mechanical properties can be characterized only by a probabilistic point of view. In particular, the goal of the proposed approach is the evaluation of the probability density function (pdf) of a single response quantity avoiding the onerous operation of the variable saturation, which is necessary when the classical PTM is applied.

1. Introduction

This work is devoted to considering some random problems where non linear transformations are involved. In particular, referring to structural systems, uncertainty on the material defines non linear transformations: that is, the relation between the structural response and the random inputs is nonlinear. This implies that the response is always non-Gaussian even if the inputs are defined as Gaussian random variables.

Referring to the general problem of non linear transformations in the literature, besides that of the application of statistical approaches based on the Monte Carlo Simulation (MCS) method [1], other statistical approaches are those based on the closure schemes on cumulants or quasi-moments, etc. [2]. Unfortunately, these approaches lose accuracy when the response is strongly non-Gaussian [3].

In the specific field of uncertain systems a very large number of papers were published in the last two decades, dealing with different approaches. The simplest methods to estimate the random characteristics of the response are the statistical approaches based on the MCS [4,5], but they are the worst methods from a computational point of view. For this reason, some alternative non-statistical methods have been proposed, as the perturbation approaches [6,7], whose drawback lies in the consistent loss of accuracy when the level of uncertainty of the structural parameters increases. These problems remain even if some efforts have been made in order to improve the approach [8]. Other non-statistical approaches are based on the expansion methods of the structural stiffness matrix in order to perform explicitly its inversion [9,10]. Another important class of methods for solving uncertain structural systems is that of projection approaches; among these, one of the most used is that based on the polynomial chaos expansion [11].

The common characteristic of all these non-statistical probabilistic approaches is that the probabilistic description of the response is related to the knowledge of its statistical moments, cumulants, correlations or power spectral densities. In any case, these quantities have dimensions of much greater order than that of the response dimensions. Moreover, if the effective non-Gaussianity of the response is considered, it is easy to realize that the stochastic analysis of these systems is a very heavy problem from a computational point of view, the evaluation of statistics of greater order than two being necessary. This drawback is above all emphasized in the field of the reliability analysis where the response probabilistic description is required directly on the probability density function (pdf) of a single component of the response.

Recently a non-statistical non-perturbative MCS method has been proposed for the analysis of FE discretized uncertain linear structures [12,13].

More recently, a new approach for the analysis of random systems has been introduced with the aim to overcome the difficulties previously described [14,15]. This method can be considered as a new version of PTM [16]: it allows the defining of some integral or differential relations that provide a direct link between the probability density of input and output. These relations, in some cases can be solved in closed form, as shown in [14,15] for some problems associated with linear and non-linear transformations. In this work, this method will be applied to study systems with uncertainties, which define a particular class of nonlinear transformations.

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2. The probabilistic transformation method

The basic aspects of the proposed approach must be researched in the theory of the space transformation of random vectors, briefly discussed below. This transformation allows making clear the direct relationship between two random vectors in terms of their pdfs.

Let $\mathbf{x}$ be an n-dimensional random vector with joint pdf $p_{\mathbf{x}}(\mathbf{x})$ and $\mathbf{h}(\bullet)$ be an n-dimensional invertible application, such that $\mathbf{h}^{-1}(\bullet) = \mathbf{f}(\bullet)$; then it is possible to write

$$\mathbf{z} = \mathbf{h}(\mathbf{x}); \quad \mathbf{x} = \mathbf{f}(\mathbf{z}) \tag{1a, b}$$

It is well known that the pdfs of these two random vectors $\mathbf{x}$ and $\mathbf{z}$, are related as follows [17]:

$$p_{\mathbf{z}}(\mathbf{z}) = \frac{1}{|\det[\mathbf{J}_{\mathbf{h}}(\mathbf{f}(\mathbf{z}))]|} p_{\mathbf{x}}(\mathbf{f}(\mathbf{z})) \tag{2}$$

where $\mathbf{J}_{\mathbf{h}}(\mathbf{z})$is the Jacobian matrix related to the transformations given in Eq. (1). Eq. (2) allows the determining of the pdf $p_{\mathbf{z}}(\mathbf{z})$, once the pdf $p_{\mathbf{x}}(\mathbf{x})$ and the transformation law are given, and it represents the reference *differential* relationship of the PTM.

The direct application of the PTM to structural problems can show the following drawbacks:

1. The orders of $\mathbf{x}$ and $\mathbf{z}$ must be equal.
2. The evaluation of $\mathbf{f}(\bullet) = \mathbf{h}^{-1}(\bullet)$ is often a very hard task.
3. The extraction of the marginal pdf $p_{z_i}(z_i)$ from the joint pdf $p_{\mathbf{z}}(\mathbf{z})$ may be computationally heavy.

The first drawback is not a restriction of the method because it is possible to make the vectors of the same order by introducing some auxiliary variables [18]. The second and third drawbacks can be, in some cases, avoided by applying the new version of the PTM, which represents the aim of this work.

From Eq. (2) can be derived an important integral relation, which is also able to establish a direct link between the probability density function of $\mathbf{x}$ and $\mathbf{z}$. In order to show this, Eq. (2) is rewritten in the following way:

$$p_{\mathbf{z}}(\mathbf{z}) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \frac{1}{|\det[\mathbf{J}_{\mathbf{h}}(\mathbf{y})]|} p_{\mathbf{x}}(\mathbf{y}) \delta(\mathbf{y} - \mathbf{f}(\mathbf{z})) \mathrm{d}y_1 \cdots \mathrm{d}y_n \tag{3}$$

In Eq. (3) the multi-dimensional Dirac Delta, centered in the point $\mathbf{y} = \mathbf{f}(\mathbf{z})$, was introduced

$$\delta(\mathbf{y} - \mathbf{f}(\mathbf{z})) = \delta(y_1 - f_1(\mathbf{z}))\delta(y_2 - f_2(\mathbf{z})).....\delta(y_n - f_n(\mathbf{z})) \tag{4}$$

The multi-dimensional Dirac Delta introduced above has non-zero value only if $\mathbf{y} = \mathbf{f}(\mathbf{z})$; then, it is equivalent to a multi-dimensional Dirac Delta centered in $\mathbf{z} = \mathbf{f}^{-1}(\mathbf{y}) = \mathbf{h}(\mathbf{y})$, provided to introduce the determinant of the Jacobian matrix related to the application $\mathbf{h}(\bullet)$, that is

$$\delta(\mathbf{y} - \mathbf{f}(\mathbf{z})) = |\det[\mathbf{J}_{\mathbf{h}}(\mathbf{y})]| \delta(\mathbf{z} - \mathbf{h}(\mathbf{y})) \tag{5}$$

It is simple to verify that the determinant of the Jacobian matrix $\mathbf{J}_{\mathbf{h}}$ assures that the functions appearing in both sides of Eq. (5) have area equal to 1. It is important to note that the validity of Eq. (5) is limited to a biunique functions; nevertheless, this is not a restriction, and a generalization can be shown. However here it is omitted in order not to make the exposition of the method heavy.

After the insertion of the last equation into Eq. (3), the following integral relationship is obtained:

$$p_{\mathbf{z}}(\mathbf{z}) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} p_{\mathbf{x}}(\mathbf{y}) \delta(\mathbf{z} - \mathbf{h}(\mathbf{y})) \mathrm{d}y_1 \cdots \mathrm{d}y_n \tag{6}$$

As said before, the random response variable of interest may be only one, but it is related to all the random input variables collected into the vector $\mathbf{x}$. Let z_j be the random response variable of interest, related to the n-vector $\mathbf{x}$ by the scalar transformation $h_j(\bullet)$ (j-th element of the application $\mathbf{h}(\bullet)$), that is

$$z_j = h_j(\mathbf{x}) \tag{7}$$

Then, by integrating both sides of Eq. (6) with respect to all the variables $z_i, i = 1, 2, 3, ..., n; i \neq j$, the following relationship is obtained:

$$p_{z_j}(z_j) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} p_{\mathbf{x}}(\mathbf{y}) \delta(z_j - h_j(\mathbf{y})) \mathrm{d}y_1 \cdots \mathrm{d}y_n \tag{8}$$

This, as will be seen in the following sections, is very suitable for defining the response pdf.

Equation of the type given into Eq. (6) will be advantageously used in the next sections of this work, and are the reference *integral* relationships of the new version of the PTM.

The difficulty in applying the Eqs. (6) and (8) depends on the transformation $h_j(\bullet)$. It is useful to define the following classification:

(a) One defines linear transformations all those transformations so that $h_j(\bullet)$ is a linear applications. This category includes all linear systems subject to random forcing.
(b) One defines nonlinear transformations all those transformations for which $h_j(\bullet)$ is not linear.

This category includes the following cases:

(b1) Linear structural systems subject to loads defined as a nonlinear function of random variables.
(b2) Linear structural systems with uncertain parameters.
(b3) Structural systems with non-linear constitutive properties subject to random forcing and/or with uncertain parameters.

The cases (a) and (b1) have been addressed in [14,15] where some explicit and approximated solutions based on the application of the PTM were given. In this paper we will address the case (b2).

It is worthy to note that the new version of PTM is not a revolution of the stochastic analysis, but the introduction of a new philosophy of stochastic analysis that, still in its infancy, is trying to overcome the limitations of known approaches.

3. Uncertain structural systems

The response of a discretized structural system, whose geometrical and/or mechanical properties are not deterministic, but defined from a probabilistic point of view, is governed by an equilibrium equation that can be expressed in the following form:

$$\mathbf{K}(\boldsymbol{\alpha})\mathbf{u}(\boldsymbol{\alpha}) = \mathbf{F} \tag{9}$$

where $\boldsymbol{\alpha}$ is the m-vector collecting the random uncertain parameters of the structural system, that is supposed to be defined through the knowledge of the joint pdf $p_{\boldsymbol{\alpha}}(\boldsymbol{\alpha})$; $\mathbf{K}$ is the structural stiffness $n \times n$ matrix depending on the uncertain parameters; $\mathbf{F}$ is the n-vector of the external actions, here considered deterministic, and $\mathbf{u}$ is the n-vector of the response displacements, depending on the structural parameters, and hence on $\boldsymbol{\alpha}$, apart from on the external actions.

In order to express the relationship between $\boldsymbol{\alpha}$ and $\mathbf{u}$ in the form given in Eq. (1a), the inversion of Eq. (9) is required, giving

$$\mathbf{u} = \mathbf{K}^{-1}(\boldsymbol{\alpha})\mathbf{F} \tag{10}$$

The specification of the fundamental relationship given by Eq. (2) to this relationship gives

$$p_{\mathbf{u}}(\mathbf{u}) = \frac{1}{|\det[\mathbf{J}(\mathbf{u})]|} p_{\boldsymbol{\alpha}}(\boldsymbol{\alpha}(\mathbf{u})) \tag{11}$$