Probabilistic Engineering Mechanics 24 (2009) 278–287

Contents lists available at ScienceDirect

Probabilistic Engineering Mechanics

journal homepage: www.elsevier.com/locate/probengmech

Stochastic sensitivity analysis by dimensional decomposition and score functions

Sharif Rahman *

College of Engineering, The University of Iowa, Iowa City, IA 52242, United States

ARTICLE INFO

Article history:
Received 11 August 2007
Received in revised form
22 June 2008
Accepted 24 July 2008
Available online 31 July 2008

Keywords:
Gradients
Reliability
Reliability-based design
Robust design
Optimization
Monte Carlo simulation

ABSTRACT

This article presents a new class of computational methods, known as dimensional decomposition methods, for calculating stochastic sensitivities of mechanical systems with respect to probability distribution parameters. These methods involve a hierarchical decomposition of a multivariate response function in terms of variables with increasing dimensions and score functions associated with probability distribution of a random input. The proposed decomposition facilitates univariate and bivariate approximations of stochastic sensitivity measures, lower-dimensional numerical integrations or Lagrange interpolations, and Monte Carlo simulation. Both the probabilistic response and its sensitivities can be estimated from a single stochastic analysis, without requiring performance function gradients. Numerical results indicate that the decomposition methods developed provide accurate and computationally efficient estimates of sensitivities of statistical moments or reliability, including stochastic design of mechanical systems. Future effort includes extending these decomposition methods to account for the performance function parameters in sensitivity analysis.

1. Introduction

Sensitivity analysis provides an important insight about complex model behavior [1,2] so that one can make informed decisions on minimizing the variability of a system [3], or optimizing a system's performance with an acceptable risk [4]. For estimating the derivative or sensitivity[1] of a general probabilistic response, there are three principal classes of methods or analyses. The *finite-difference method* [5] involves repeated stochastic analyses for nominal and perturbed values of system parameters, and then invoking forward, central, or other differentiation schemes to approximate their partial derivatives. This method is cumbersome and often expensive, if not prohibitive, because evaluating probabilistic response for each system parameter, which constitutes a complete stochastic analysis, is already a computationally demanding task. The two remaining methods, the *infinitesimal perturbation analysis* [6,7] and the *score function method* [8], have been mostly viewed as competing methods, where both performance and sensitivities can be obtained from a single stochastic simulation. However, there are additional requirements of regularity conditions, in particular smoothness of the performance function or the probability measure [9]. For the infinitesimal perturbation analysis, the probability measure is fixed, and the derivative of a performance function is taken, assuming that the differential and integral operators are interchangeable. The score function method, which involves probability measure that continuously varies with respect to a design parameter, also requires a somewhat similar interchange of differentiation and integration, but in many practical examples, interchange in the score function method holds in a much wider range than that in infinitesimal perturbation analysis. Nonetheless, both methods, when valid, are typically employed in conjunction with the *direct* Monte Carlo simulation, a premise well-suited to stochastic optimization of discrete event systems. Unfortunately, in mechanical design optimization, where stochastic response and sensitivity analyses are required at each design iteration, even a single Monte Carlo simulation is impractical, as each deterministic trial of the simulation may require expensive finite-element or other numerical calculations. This is the principal reason why neither the infinitesimal perturbation analysis nor the score function method have found their way in to the design optimization of mechanical systems.

The direct differentiation method, commonly used in deterministic sensitivity analysis [10], provides an attractive alternative to the finite-difference method for calculating stochastic sensitivities. In conjunction with the first-order reliability method, Liu and Der Kiureghian [11] and their similar work has significantly contributed to the development of such methods for obtaining reliability sensitivities. The direct differentiation method, also capable of generating both reliability and its sensitivities from a single stochastic analysis, is particularly effective in solving finite-element-based reliability problems, when (1) the most probable point can be efficiently located and (2) a linear approximation of the performance function at that point is adequate. Therefore, the

* Corresponding address: Department of Mechanical & Industrial Engineering, The University of Iowa, Iowa City, IA 52242, United States. Tel.: +1 319 335 5679; fax: +1 319 335 5669.
E-mail address: rahman@engineering.uiowa.edu.

[1] The nouns "derivative" and "sensitivity" are used synonymously in this paper.

0266-8920/$ – see front matter © 2008 Elsevier Ltd. All rights reserved.
doi:10.1016/j.probengmech.2008.07.004

direct differentiation method inherits high efficiency of the first-order reliability method, but also its limitations. In contrast, the three sensitivity methods described in the preceding are independent of underlying stochastic analysis.

This article presents a new class of computational methods, known as dimensional decomposition methods, for calculating stochastic sensitivities of mechanical systems with respect to probability distribution parameters. The idea of dimensional decomposition of a multivariate function, originally developed by the author's group for statistical moment [12,13] and reliability [14] analyses, has been extended to stochastic sensitivity analysis, which is the focus of the current paper. Section 2 describes a unified probabilistic response and sensitivity, and derives score functions associated with a number of probability distributions. Section 3 presents the dimensional decomposition method for calculating the probabilistic sensitivities, using either the numerical integration or the simulation method and score functions. The computational effort required by the decomposition method is also discussed. Four numerical examples illustrate the accuracy, computational efficiency, and usefulness of the sensitivity method in Section 4. Section 5 states the limitations of the proposed method. Finally, conclusions are drawn in Section 6.

2. Probabilistic response and sensitivity

Let $(\Omega, \mathcal{F})$ be a measurable space, where Ω is a sample space and $\mathcal{F}$ is a σ-field on Ω. Defined over $(\Omega, \mathcal{F})$, consider a family $\{P_{\boldsymbol{\theta}} : \mathcal{F} \to [0,1]\}$ of probability measures, where $\boldsymbol{\theta} = \{\theta_1, \ldots, \theta_M\}^T \in \mathbb{R}^M$ is an M-dimensional vector of deterministic parameters and $\mathbb{R}^M$ is an M-dimensional, real, vector space. In other words, a sample point $\omega \in \Omega$ obeys the probability law $P_{\boldsymbol{\theta}}(F)$ for any event $F \in \mathcal{F}$ and $\boldsymbol{\theta} \in \mathbb{R}^M$, so that the probability triple $(\Omega, \mathcal{F}, P_{\boldsymbol{\theta}})$ depends on $\boldsymbol{\theta}$.

Let $\{\boldsymbol{X} = \{X_1, \ldots, X_N\}^T : (\Omega, \mathcal{F}) \to (\mathbb{R}^N, \mathcal{B}^N)\}$ with $\mathcal{B}^N$ as the Borel σ-field on $\mathbb{R}^N$ denote a family of $\mathbb{R}^N$-valued input random vector, which describes statistical uncertainties in loads, material properties, and geometry of a mechanical system. The probability law of $\boldsymbol{X}$ is completely defined by a family of joint probability density functions $\{f_{\boldsymbol{X}}(\boldsymbol{x}; \boldsymbol{\theta}), \boldsymbol{x} \in \mathbb{R}^N, \boldsymbol{\theta} \in \mathbb{R}^M\}$ that are associated with probability measures $\{P_{\boldsymbol{\theta}}, \boldsymbol{\theta} \in \mathbb{R}^M\}$. Let $y(\boldsymbol{X})$, a real-valued, measurable transformation on $(\Omega, \mathcal{F})$, define a relevant performance function of a mechanical system. It is assumed that $y : (\mathbb{R}^N, \mathcal{B}^N) \to (\mathbb{R}, \mathcal{B})$ is not an explicit function of $\boldsymbol{\theta}$, although y implicitly depends on $\boldsymbol{\theta}$ via the probability law of $\boldsymbol{X}$. The objective of stochastic sensitivity analysis is to obtain the partial derivatives of a probabilistic characteristic of $y(\boldsymbol{X})$ with respect to a parameter θ_i, $i = 1, \ldots, M$, given a reasonably arbitrary probability law of $\boldsymbol{X}$.

2.1. Statistical moments and reliability

Denote by $\mathcal{L}_q(\Omega, \mathcal{F}, P_{\boldsymbol{\theta}})$ a collection of real-valued random variables including $y(\boldsymbol{X})$, which is defined on $(\Omega, \mathcal{F}, P_{\boldsymbol{\theta}})$ such that $\mathbb{E}[|y^q(\boldsymbol{X})|] < \infty$, where $q \geq 1$ is an integer and $\mathbb{E}_{\boldsymbol{\theta}}$ represents the expectation operator with respect to the probability measure $\{P_{\boldsymbol{\theta}}, \boldsymbol{\theta} \in \mathbb{R}^M\}$. If $y(\boldsymbol{X})$ is in $\mathcal{L}_q(\Omega, \mathcal{F}, P_{\boldsymbol{\theta}})$, then its qth moment, defined by the multifold integral

$$m_q(\boldsymbol{\theta}) := \mathbb{E}_{\boldsymbol{\theta}}\left[y^q(\boldsymbol{X})\right] := \int_{\mathbb{R}^N} y^q(\boldsymbol{x}) f_{\boldsymbol{X}}(\boldsymbol{x}; \boldsymbol{\theta}) \mathrm{d}\boldsymbol{x}; \quad q = 1, 2, \ldots, \tag{1}$$

exists and is finite. A similar integral appears in time-invariant reliability analysis, which entails calculating the failure probability

$$P_F(\boldsymbol{\theta}) := P_{\boldsymbol{\theta}}[\boldsymbol{X} \in \Omega_F] = \int_{\mathbb{R}^N} I_{\Omega_F}(\boldsymbol{x}) f_{\boldsymbol{X}}(\boldsymbol{x}; \boldsymbol{\theta}) \mathrm{d}\boldsymbol{x} := \mathbb{E}_{\boldsymbol{\theta}}\left[I_{\Omega_F}(\boldsymbol{X})\right], \tag{2}$$

where $\Omega_F := \{\boldsymbol{x} : y(\boldsymbol{x}) < 0\}$ is the failure set for component reliability analysis; and $\Omega_F := \{\boldsymbol{x} : \cup_{k=1}^K y^{(k)}(\boldsymbol{x}) < 0\}$ and $\Omega_F := \{\boldsymbol{x} : \cap_{k=1}^K y^{(k)}(\boldsymbol{x}) < 0\}$ are the failure sets for series-system and parallel-system reliability analyses, respectively, with $y^{(k)}(\boldsymbol{x})$ representing the kth out of K performance functions, and

$$I_{\Omega_F}(\boldsymbol{x}) := \begin{cases} 1, & \boldsymbol{x} \in \Omega_F \\ 0, & \boldsymbol{x} \in \Omega \setminus \Omega_F \end{cases}; \quad \boldsymbol{x} \in \mathbb{R}^N \tag{3}$$

is an indicator function. Therefore, expressions of both integrals or expectations in Eqs. (1) and (2) can be consolidated into a generic probabilistic response

$$h(\boldsymbol{\theta}) = \mathbb{E}_{\boldsymbol{\theta}}[g(\boldsymbol{X})] := \int_{\mathbb{R}^N} g(\boldsymbol{x}) f_{\boldsymbol{X}}(\boldsymbol{x}; \boldsymbol{\theta}) \mathrm{d}\boldsymbol{x}, \tag{4}$$

where $h(\boldsymbol{\theta})$ and $g(\boldsymbol{x})$ are either $m_q(\boldsymbol{\theta})$ and $y^q(\boldsymbol{x})$, respectively, for statistical moment analysis or $P_F(\boldsymbol{\theta})$ and $I_{\Omega_F}(\boldsymbol{x})$, respectively, for reliability analysis.

2.2. Sensitivity analysis by score functions

Consider a distribution parameter $\theta_i, i = 1, \ldots, M$, and suppose that the derivative of a generic probabilistic response $h(\boldsymbol{\theta})$, which is either the statistical moment of a mechanical response or the reliability of a mechanical system, with respect to θ_i is sought. For such sensitivity analysis, the following assumptions are required [8].

1. The probability density function $f_{\boldsymbol{X}}(\boldsymbol{x}; \boldsymbol{\theta})$ is continuous. Discrete distributions having jumps at a set of points, or a mixture of continuous and discrete distributions, can be treated similarly, but will not be discussed here.
2. The parameter $\theta_i \in \Theta_i \subset \mathbb{R}$, $i = 1, \ldots, M$, where Θ_i is an open interval on $\mathbb{R}$.
3. The partial derivative $\partial f_{\boldsymbol{X}}(\boldsymbol{x}; \boldsymbol{\theta})/\partial\theta_i$ exists and is finite for all $\boldsymbol{x}$ and $\theta_i \in \Theta_i \subset \mathbb{R}$. In addition, $h(\boldsymbol{\theta})$ is a differentiable function of $\boldsymbol{\theta} \in \mathbb{R}^M$.
4. There exists a Lebesgue integrable dominating function $r(\boldsymbol{x})$ such that
$$\left| g(\boldsymbol{x}) \frac{\partial f_{\boldsymbol{X}}(\boldsymbol{x}; \boldsymbol{\theta})}{\partial\theta_i} \right| \leq r(\boldsymbol{x}) \tag{5}$$
for all $\boldsymbol{\theta} \in \mathbb{R}^M$.

The assumptions 1–4 are known as the regularity conditions.

Taking the partial derivative of both sides of Eq. (4) with respect to θ_i gives

$$\frac{\partial h(\boldsymbol{\theta})}{\partial\theta_i} = \frac{\partial}{\partial\theta_i} \int_{\mathbb{R}^N} g(\boldsymbol{x}) f_{\boldsymbol{X}}(\boldsymbol{x}; \boldsymbol{\theta}) \mathrm{d}\boldsymbol{x}. \tag{6}$$

By invoking assumption 4 and the Lebesgue dominated convergence theorem [15], the differential and integral operators can be interchanged, yielding

$$\begin{aligned} \frac{\partial h(\boldsymbol{\theta})}{\partial\theta_i} &= \int_{\mathbb{R}^N} g(\boldsymbol{x}) \frac{\partial f_{\boldsymbol{X}}(\boldsymbol{x}; \boldsymbol{\theta})}{\partial\theta_i} \mathrm{d}\boldsymbol{x} \\ &= \int_{\mathbb{R}^N} g(\boldsymbol{x}) \frac{\partial \ln f_{\boldsymbol{X}}(\boldsymbol{x}; \boldsymbol{\theta})}{\partial\theta_i} f_{\boldsymbol{X}}(\boldsymbol{x}; \boldsymbol{\theta}) \mathrm{d}\boldsymbol{x} \\ &= \mathbb{E}_{\boldsymbol{\theta}}\left[g(\boldsymbol{X}) \frac{\partial \ln f_{\boldsymbol{X}}(\boldsymbol{X}; \boldsymbol{\theta})}{\partial\theta_i}\right]; \quad i = 1, \ldots, M, \end{aligned} \tag{7}$$

provided $f_{\boldsymbol{X}}(\boldsymbol{x}; \boldsymbol{\theta}) \neq 0$. Define

$$s_{\theta_i}^{(1)}(\boldsymbol{x}; \boldsymbol{\theta}) := \frac{\partial \ln f_{\boldsymbol{X}}(\boldsymbol{x}; \boldsymbol{\theta})}{\partial\theta_i}, \tag{8}$$

which is known as the first-order score function for the parameter θ_i [8]. Therefore, the first-order sensitivity of $h(\boldsymbol{\theta})$ can be expressed by

$$\frac{\partial h(\boldsymbol{\theta})}{\partial\theta_i} = \mathbb{E}_{\boldsymbol{\theta}}\left[g(\boldsymbol{X}) s_{\theta_i}^{(1)}(\boldsymbol{X}; \boldsymbol{\theta})\right]; \quad i = 1, \ldots, M. \tag{9}$$