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Estimation of minimum cross-entropy quantile function using fractional probability weighted moments

Jian Deng[a,b,*], M.D. Pandey[a]

[a] Department of Civil Engineering, University of Waterloo, Waterloo, Ontario, N2L 3G1, Canada
[b] School of Resources and Safety Engineering, Central South University, Changsha, 410083, China

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Abstract

The principle of minimum cross-entropy provides a systematic approach to derive the posterior distribution of a random variable given a prior and additional information in terms of its product moments. This approach can be extended to derive directly the quantile function by using probability weighted moments (PWMs) as constraints in the cross-entropy minimization approach, as shown in a previous study [Pandey MD. Extreme quantile estimation using order statistics with minimum cross-entropy principle. Probabilistic Engineering Mechanics 2001;16(1):31–42]. The objective of the present paper is to extend and improve the previous method by incorporating the use of the fractional probability weighted moments (FPWMs) in the place of conventional integer-order PWMs. A new and general estimation method is proposed in which the Monte Carlo simulations and optimization algorithms are combined to estimate FPWMs that would subsequently lead to the best-fit quantile function. The numerical examples presented in the paper show a substantial improvement in accuracy by the use of the proposed method over the conventional approach.

Keywords: Probability weighted moment; Minimum cross-entropy principle; Quantile function; Generalized Pareto distribution; Weibull distribution; Extreme value analysis

1. Introduction

In traditional methods of statistical inference, the probability distribution of a random variable is judged empirically from the available data, and then distribution parameters are estimated using methods such as Maximum Likelihood method. However, the bias and efficiency of quantile estimates remain sensitive to the type of assumed distribution. An alternative approach to the probabilistic inference comes from the modern information theory in which the concept of entropy was introduced.

The Maximum Entropy principle (MaxEnt) was presented as a rational approach for choosing the least biased probability distribution among all possible distributions which are consistent with available data and contain the minimum amount of spurious information [2–6]. Later, a refined method was developed to integrate a prior distribution with available data for inference purposes based on the concept of cross-entropy [7–9]. The minimum cross-entropy principle (CrossEnt) states that for a prior distribution and some data, select that posterior distribution which minimizes the cross-entropy. When only product moment constraints are specified, the cross-entropy minimization satisfies all consistency axioms [8]. However, the estimates of higher-order product moments (order >2) from small samples (less than 30) tend to have large sampling error. The CrossEnt and/or MaxEnt distribution derived from poor moment estimates would lead to inaccurate quantile values. This difficulty can be circumvented by using the probability weighted moments (PWMs) in the place of ordinary moments, as shown by Pandey [1].

PWMs are widely used in many areas of science and engineering [10–13]. The main advantage of using PWMs is that their higher-order values can be accurately estimated from small samples. Also, PWMs are shown to be fairly insensitive

* Corresponding author at: School of Resources and Safety Engineering, Central South University, Changsha, 410083, China. Tel.: +1 519 8884567 35900.
E-mail addresses: jiandeng@mail.csu.edu.cn (J. Deng), mdpandey@uwaterloo.ca (M.D. Pandey).

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to outliers (extreme observations) in the data [14], because they are linear combinations of the observed data values, unlike the ordinary statistical moments, where the data are squared, cubed, etc.

In the previous study [1], PWMs were used to derive directly the quantile function (QF) of a random variable. In this paper, we extend and improve the method by using fractional PWMs in the place of PWMs of integer order. The fractional PWMs are analogous to the product moments of fractional order, which are discussed in a series of recent papers [15–17]. This paper shows that the proposed FPWM-based method is effective in modeling the distribution tail and estimating extreme quantiles from a small sample of data.

2. Probability weighted moments (PWMs)

2.1. Definitions

The PWM of a random variable was formally defined by Greenwood et al. [10] as

$$M_{r,s,t} = E[X^r F^s (1-F)^t] = \int_0^1 [x(F)]^r F^s (1-F)^t \mathrm{d}F$$

where X denotes the random variable and F denotes the probability. In an analogy with the usual product moments, r, s, t are taken as integers in most applications of PWMs. In such cases, we refer $M_{r,s,t}$ as integer-order probability weighted moments (IPWMs). In this paper, we define $M_{r,s,t}$ as fractional probability weighted moments (FPWMs) by taking the exponents, r, s, and t as real positive numbers. The following two forms of PWM (IPWM or FPWM) are particularly simple and useful in practical applications:

$$\text{Type 1:} \quad \alpha_t = M_{1,0,t} = \int_0^1 x(F)(1-F)^t \mathrm{d}F \tag{1}$$

and

$$\text{Type 2:} \quad \beta_s = M_{1,s,0} = \int_0^1 x(F) F^s \mathrm{d}F. \tag{2}$$

For an ordered sample, $x_1 \le x_2 \le \cdots \le x_n$, the Type 1 and Type 2 IPWMs can also be estimated as

$$a_t = \frac{1}{n}\sum_{i=1}^{n}\left[\binom{n-i}{t} x_i\right] \Big/ \binom{n-1}{t} \tag{3}$$

$$b_s = \frac{1}{n}\sum_{i=1}^{n}\left[\binom{i-1}{s} x_i\right] \Big/ \binom{n-1}{s} \tag{4}$$

where $t, s = 0, 1, \ldots, (n-1)$ are non-negative integers and the binomial coefficient is given as

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}, \quad \text{if } n \ge r \ge 0; \quad \text{others } 0.$$

FPWMs are estimated from the order statistics of a sample

$$a_t = \frac{1}{n}\sum_{i=1}^{n}[(1-P_i)^t x_i] \tag{5}$$

$$b_s = \frac{1}{n}\sum_{i=1}^{n}[P_i^s x_i] \tag{6}$$

where x_i is the ith-order statistics, $x_1 \le x_2 \le \cdots \le x_n$, and P_i is the probability plotting position of x_i computed by a suitable formula [18], such as

$$P_i = \frac{i - 0.35}{n}.$$

2.2. PWMs as moments of quantile function

For a non-negative random variable, β_s can be interpreted as a moment of the quantile function [1]. Recall the definition of an ordinary statistical moment of sth order (remember s is an integer or real number)

$$E[X^s] = \int_R x^s f(x)\mathrm{d}x = \int_0^1 [x(u)]^s \mathrm{d}u \quad \text{where } \mathrm{d}u = \mathrm{d}F(x) = \frac{f(x)\mathrm{d}x}{\int f(x)\mathrm{d}x}. \tag{7}$$

Here $\mathrm{d}u = \mathrm{d}F(x)$ is a probability measure, which is a monotonic, continuous and non-negative function with $0 \le F(x) \le 1$. Another probability measure, $\mathrm{d}T(u)$, can be introduced using the following normalizing transformation:

$$\mathrm{d}T(u) = \frac{x(u)\mathrm{d}u}{\int_0^1 x(u)\mathrm{d}u} = \frac{x(u)\mathrm{d}u}{\beta_0}$$

where β_0 $(=\alpha_0)$ is the area under the quantile function, and is also equal to the average of the random variable. Thus the PWM can be redefined using a new measure, $\mathrm{d}T$ as

$$\beta_k = \beta_0 \int_0^1 u^k \mathrm{d}T(u). \tag{8}$$

Comparing Eqs. (7) and (8), (β_k/β_0) can be interpreted as the kth moment of the quantile function, $x(u)$, $0 < u < 1$. Specifically, when s is an integer number, β_s is the IPWM of QF, and s is a real number, the β_s is the FPWM of QF. Similarly, (α_k/α_0) can be interpreted as the kth moment of $x(q)$ where $q = (1-u)$ is the probability of exceedance.

3. Minimum cross-entropy quantile function

3.1. Entropy

For a continuous random variable X, with the density function $f(x)$, the entropy is defined as

$$H[f(x)] = -\int_{-\infty}^{+\infty} f(x)[\ln f(x)]\mathrm{d}x.$$

In the context of prior and posterior probabilities, Kullback [7] introduced the concept of cross-entropy, or directed divergence, or probabilistic distance, of a posterior probability distribution $f(x)$ from a prior distribution $g(x)$:

$$S(f,g) = -\int_{-\infty}^{+\infty} f(x)\ln\left(\frac{f(x)}{g(x)}\right)\mathrm{d}x. \tag{9}$$