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Mean points of failure and safety domains: Estimation and application

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Abstract

A conceptual representation of the mean points of failure and safety domains in reliability problems is presented, and a practical approach for approximating the mean points based on the first-order reliability method (FORM) is proposed. The effectiveness of the approach is demonstrated with two nonlinear examples. The mean points of failure and safety domains and the approach for approximating them are then used to develop an innovative method for estimating transition probabilities between failure and safety domains in discrete time-variant reliability problems. The performance of the new method is demonstrated in example problems. It is shown that, from a practical viewpoint, the method is quite effective. Although the example problems are stationary univariate systems, the method shows promise for being applicable for more complex cases, such as non-stationary multivariate systems.

Keywords: Reliability; Failure domain; Safety domain; FORM; Time-variant reliability problems; Transition probabilities; Resilience

1. Introduction

In civil engineering reliability analyses, it may be useful to estimate the mean points of the failure and safety domains for a given system. For example, both points may help in estimating two-state transition probabilities between both domains for time-variant reliability problems in the discrete time domain, or their derivatives such as resilience. Estimates of the transition probabilities may be useful in determining the mean number of crossings or the mean interval between crossings in the analysis of dynamic loads on structures, or the likely length of water shortages in water resource allocation strategies.

This paper introduces definitions of the mean points of the failure and safety domains, and develops a practical approach for approximating these mean points based on the first-order reliability method (FORM). As one of the potential applications of the mean points, an innovative method of estimating two-state transition probabilities in discrete processes is then proposed based on the mean points. The method is demonstrated for simple cases constructed on a univariate Gaussian (or normal) process. The results of this application indicate that the method is effective in such cases and shows promise for accommodating general cases described by a nonlinear performance function constructed on multivariate and non-normal processes.

The remaining sections of this paper are organized as follows. In Section 2, the mean points are mathematically defined. In Section 3, first-order approximations of the mean points are developed based on FORM, and demonstrated in two nonlinear numerical examples. The application of the mean points for determining two-state transition probabilities is proposed in Section 4. Section 5 presents conclusions.

2. Definitions

Reliability is defined on the basis of a load–resistance interference featured by a performance function g, which involves a set of stochastic variables ($\mathbf{x}$) and a set of deterministic parameters. States of g are divided into two categories, safety ($g > 0$) or failure ($g \leq 0$). The boundary between both states is called the limit state surface where $g = 0$. Thus the probabilistic $\mathbf{x}$-space is divided into two domains by the two states, i.e. the failure domain F where $g \leq 0$, and the safety domain S where $g > 0$. At a given time moment, the probability of a system being in S is referred to as reliability (or

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Nomenclature

C	a constant;
E	mean;
E_F	an expectation over the failure domain;
F	the failure domain;
f	the joint PDF of random variables;
g	performance function;
n	the dimension of random variables;
P	probability;
P_F	failure probability;
P_{FS}	the one-step transition probability from F to S;
P_S	reliability;
P_{SF}	the one-step transition probability from S to F;
Re	resilience;
S	the safety (or success) domain;
t	time moment;
$\mathbf{u}$	a set of standardized uncorrelated normal variables $(u_1, u_2, \ldots, u_n)$;
$\mathbf{u_F}$	the MPFD of $\mathbf{u}$-space;
$\mathbf{u_S}$	the MPSD of $\mathbf{u}$-space;
$\mathbf{u}^*$	design point;
$\mathbf{u_F}^*$	the first-order approximation of $\mathbf{u_F}$;
$\mathbf{u_S}^*$	the first-order approximation of $\mathbf{u_S}$;
V	a stationary univariate Gaussian process;
$v(t)$	the value of V at t;
v_F	the MPFD of the v-space;
$v_F{}^*$	the first-order approximation of the MPFD of the v-space;
$\mathbf{w}$	standard normal random variables in discrete representation;
$\mathbf{x}$	a set of random variables $(x_1, x_2, \ldots, x_n)$;
$\mathbf{x_F}$	the MPFD in $\mathbf{x}$-space $(x_{F1}, x_{F2}, \ldots, x_{Fn})$;
$\mathbf{x_S}$	the MPSD in $\mathbf{x}$-space $(x_{S1}, x_{S2}, \ldots, x_{Sn})$;
y	any point in F;
z	a standard normal variable transformed from g;
z_F	the mapped value of $\mathbf{u_F}^*$ in z-space;
z_S	the mapped value of $\mathbf{u_S}^*$ in z-space;
z_β	the mapped value of $\mathbf{u}^*$ in z-space;
$\boldsymbol{\alpha}$	the negative of the normalized gradient vector of the limit state surface at $\mathbf{u}^*$;
β	reliability index;
ε	a normal white noise;
Φ	the CDF of the univariate standard normal random variable;
ϕ_k	the PDF of the k-dimensional standard normal random variable;
$\rho(1)$	the lag-1 autocorrelation coefficient of V;
$\rho_g(1)$	the lag-1 autocorrelation coefficient of g series;
σ	standard deviation;

instantaneous reliability), while the complement of reliability is known as the failure probability. Practical estimating methods for reliability and failure probability include (1) sampling methods such as those based on Monte Carlo simulations (MCS), for instance Crude Monte Carlo simulations (CMC), (2) approximations such as those based on a point estimate, or based on a reliability index for instance FORM and the second-order reliability method (SORM), and (3) combinations of above two. (Under the reliability index-based approximations, the reliability index, β, is the ratio of the mean of g to the standard deviation of g.) For more details regarding reliability methods, see [1–6].

The mean point of failure domain (MPFD), $\mathbf{x_F}$, represents the probabilistic geometric centroid of F, mathematically defined as

$$x_{Fi} = \frac{\int_F x_i f(\mathbf{x})\mathrm{d}\mathbf{x}}{\int_F f(\mathbf{x})\mathrm{d}\mathbf{x}} = \frac{\int_F x_i f(\mathbf{x})\mathrm{d}\mathbf{x}}{P_F} \quad (i = 1, 2, \ldots n) \tag{1}$$

where $\mathbf{x} = [x_1 \ldots x_n]^{\mathrm{T}}$, the vector of random variables; $f(\mathbf{x})$ = the joint probability density function (PDF) of $\mathbf{x}$; x_{Fi} = the ith coordinate of $\mathbf{x_F}$; and P_F = failure probability.

Similarly, the mean point of the safety domain (MPSD), $\mathbf{x_S}$, represents the probabilistic geometric centroid of S, mathematically defined as

$$x_{Si} = \frac{\int_S x_i f(\mathbf{x})\mathrm{d}\mathbf{x}}{\int_S f(\mathbf{x})\mathrm{d}\mathbf{x}} = \frac{\int_S x_i f(\mathbf{x})\mathrm{d}\mathbf{x}}{P_S} \quad (i = 1, 2, \ldots n) \tag{2}$$

where x_{Si} = the ith coordinate of $\mathbf{x_S}$; and P_S = reliability.

Oftentimes a joint PDF is quite difficult to derive, and the integrals involved in Eqs. (1) and (2) may be impossible to solve analytically in complex systems. Therefore, except in rare cases, $\mathbf{x_S}$ and $\mathbf{x_F}$ can only be approximated in practice. Numerical schemes and MCS may be used to approximate $\mathbf{x_S}$ and $\mathbf{x_F}$, but their computational costs could be very high. This paper introduces a practical method for approximating $\mathbf{x_S}$ and $\mathbf{x_F}$ based on FORM.

3. First-order mean points of failure and safety domains

3.1. FORM

Under FORM, the original $\mathbf{x}$-space is transformed into a standardized uncorrelated normal space, $\mathbf{u}$, with zero means and unit standard deviations. To simplify the solution, FORM replaces the original g in the $\mathbf{u}$-space with its first-order Taylor series expanded at the design point, $\mathbf{u}^*$, as shown in Fig. 1. Note that, in this paper, performance functions in both $\mathbf{x}$-space and $\mathbf{u}$-space are identically denoted as g, though they may have different forms. As a local optimum on the limit state surface of g, $\mathbf{u}^*$ is locally closest to the origin of the $\mathbf{u}$-space. If $g > 0$ at the $\mathbf{u}$ origin, β is equal to the distance from the $\mathbf{u}$ origin to the limit state surface, or the norm of $\mathbf{u}^*$; otherwise, β is the negative of that distance. Because $\mathbf{u}^*$ is a local optimum, the following relationship holds:

$$\mathbf{u}^* = \beta\boldsymbol{\alpha} \tag{3}$$