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# Arithmetic and geometric solutions for average rigid-body rotation ☆

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Several existing formulations for the rotation average are reviewed and classified into the Euclidean and Riemannian solutions. A novel, more efficient characterization of the Riemannian-based average is proposed. The discussion addresses the issue of bi-invariance of the underlying distance metrics, and how the different solutions are interrelated. A not bi-invariant arithmetic average of rotation vectors is considered and shown to be an approximate solution to both the Riemannian and Euclidean averages. Results for four numerical examples are presented demonstrating the closeness of all solutions in practical applications, but also their differences when the rotations to be averaged are orthogonal to each other.



## 1. Introduction

### *1.1. Background and motivation*

The need to calculate an average of several rigid-body rotations arises in a number of applications. In robotics, for example, the ubiquitous use of cameras and their low cost make it practical to equip robotic systems with multiple cameras. These may be used to determine the pose of objects in the environment or the pose of robot end-effector and provide multiple measurements of the same. Another use of rotation averaging and more generally orientation *statistics*, is illustrated in Ref. [1], the authors of which apply their statistical approach to analyze human upper limb poses in a drilling task.

Our motivation for investigating the present subject arose from the research in human gait analysis. In this context, researchers usually collect measurements with a motion capture (MOCAP) system which generates 3D positions of markers mounted on the subject's body [2–4]. The data is then post-processed with essentially an inverse kinematics algorithm to reconstruct the joint kinematics of the body from the measured marker coordinates. One complicating factor in this procedure is the soft tissue artifact: it corrupts the validity of the rigid-body approximation to the motion of the markers. A number of algorithms have been proposed to deal with this specific problem [5–7]. One possible approach is to use *patches* of markers, the motions of which individually best match that of a rigid-body. After extracting the pose information for the patches, one would need to average the rotations from several patches on a single body segment, to obtain the best estimate of the segment's rotation. Note that depending on the particulars of the post-processing algorithm, the orientation component of the calculated pose may be represented via any of the existing rotational representations; common examples are rotation matrices, quaternions and rotation vectors.

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### 1.2. Previous Work

Only a few publications exist devoted specifically to the subject of *averaging* rotations [8–10] and we will present the corresponding formulations in detail in Section 2 of this manuscript. In [8], motivated by the sensor-fusion applications in Virtual Reality, two procedures are discussed for averaging rotations: based on the rotation matrices and based on quaternions, the latter advocated for averaging two quaternions only. A more recent reference [9] introduces the problem in the context of robot vision, mentioned earlier, where the orientation of the object is measured with multiple cameras, and also, the problem of registration of medical images. The objective in [9] is to show that the barycentric (or arithmetic) means of rotations, defined based on rotation matrix and quaternion representations, approximate the corresponding means based on the Riemannian metric. The latter is stated as $\phi(\cdot)$ and is the angle of rotation induced by the rotation matrix or the quaternion argument. This metric, further discussed in Section 2, has been used in [11] for computing the mean rotation, in [12] for measuring the distance between two rotation matrices and in [13] to determine the rotation distance between contacting polyhedra. The authors of [10] define two *bi-invariant* metrics for rotation matrices, the notion of bi-invariance to be explained shortly. These are then used as the bases for formulating the corresponding rotation means, referred to as Euclidean and Riemannian, respectively, terms which we use here interchangeably with arithmetic and geometric. The realizations of the two rotation matrix means are derived in [10], specifically, a closed-form solution in case of the Euclidean mean and a set of nonlinear equations to be solved for the Riemannian rotation matrix mean.

A related problem that has received significantly more attention, particularly in the computer graphics and robotics communities, is the problem of *interpolation* of rotations. In this context, one is looking to generate a smooth curve in time, denoted generically as $\mathbf{R}(t)$, which interpolates a specified sequence of rotations at particular time instances. The interpolations can then be used to produce, for example, a smooth motion of a robot end-effector or a camera. In the computer graphics community, use of quanternions for animating rotations has been widely popularized and researched with a number of quaternion-based spline interpolations proposed [14–17]. Other parametrizations of rotation, such as using canonical coordinates [12,17], Cayley-Rodrigues parameters [12] and Euler angles [16] have also been considered for interpolation on the space of orientations. One of the principal issues in rotation interpolation is to develop computationally efficient algorithms [17] which provide sufficiently accurate approximations to the *optimal* interpolation. The latter is typically characterized by the minimum angular acceleration of the resulting curve $\mathbf{R}(t)$, and it produces a smooth interpolated motion.

Central to both the formulation of rotation average and interpolation of rotations is the notion of the underlying *distance measure*, or metric, already mentioned above. This notion must be clearly defined when measuring a distance between two rotations because rotations are not members of a vector space, but belong to $SO(3)$, the special orthogonal group in $\Re^3$. Whichever definition for the metric one proposes, ideally we require that it be *bi-invariant*. This means that if we define a metric between two members of the group of rotations, say $d(\mathbf{R}_1,\mathbf{R}_2)$ denoting the distance between two rotation matrices $\mathbf{R}_1$ and $\mathbf{R}_2$, then it must produce the same measure when evaluated for the pair $(\mathbf{P}\mathbf{R}_1\mathbf{Q},\mathbf{P}\mathbf{R}_2\mathbf{Q})$ for every $\mathbf{P}$ and $\mathbf{Q}$ in $SO(3)$. In the context of rotation interpolation, the same bi-invariance property is critical as it requires the orientation curve to be independent of how one selects either the fixed or the moving reference frames [12], when defining the body orientation as a function of time. The notion of appropriate distance metrics on a special *Euclidean* group, $SE(3)$, has been discussed extensively for measuring the distance between *general* rigid-body displacements, particularly in the context of robot trajectory planning [18] and mechanism design [19]. It has been long established that unlike the space of orientations, no bi-invariant metric can be constructed on $SE(3)$ [20], although several propositions for left-invariant [20,21] and frame invariant (objective) solutions [22] have been made.

Having chosen the metric, one can then formulate the corresponding rotation average as the least-squares solution to the corresponding metric-based optimization problem. We note that although bi-invariance is an intuitive and meaningful requirement, it will be shown in this paper that other possible definitions of rotation average are not based on a bi-invariant metric; yet, they can produce excellent estimates of mean rotation. We also suggest that bi-invariance in the sense defined here is possible only for those rotational representations which allow for a multiplicative composition of rotations, or more precisely belong to a multiplicative group.

### 1.3. About this paper

The present manuscript is organized as follows. In Section 2 we review the existing bi-invariant formulations of the average rotation problem, based primarily on Refs. [8–10], while also referring to their use in literature, and establish clear links between the different solutions for the mean rotation. We then develop a new algebraic realization for the average rotation vector based on the aforementioned Riemannian metric $\phi(\cdot)$. Section 3 is allocated to the presentation and discussion of a non-invariant rotation average, computed as the arithmetic average of rotation vectors. It is included here because averaging of rotation vectors provides a fast and simple to implement solution for the average rotation which, somewhat unexpectedly, is close to the bi-invariant results. It is demonstrated, however, that arithmetic average of rotation vectors is an approximate solution to the Riemannian and Euclidean rotation averages. In Section 4, a summary of the existing and proposed algorithms is presented and their performance evaluated by means of four examples with the average rotations calculated by using solutions from Sections 2 and 3. The examples are comprised of three simulated test-cases and one case where the rotations to be averaged were obtained from pendulum experiments and a marker-based pose measurement system.