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A dual-level method of fundamental solutions for three-dimensional exterior high frequency acoustic problems

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ABSTRACT

Physical essence of the fictitious boundary of the method of fundamental solutions has been a mystery for a long time. In this study, we attempt to explain the reason why fictitious boundary has such a dramatic effect on numerical results. The influence law of the fictitious boundary on numerical results is revealed. Based on this understanding, a dual-level method of fundamental solutions with self-adaptive adjustment coefficients is proposed. The competitive attributes of the method are that it inherits the high numerical efficiency of the method of fundamental solutions, and it improves numerical stability significantly. The effect of the fictitious boundary on numerical results is eliminated by introducing the concept of equivalent slope. It should be noted that the dual-level method of fundamental solutions can simulate exterior high frequency acoustic problems under the lowest sampling frequency allowed by the Shannon's sampling theorem. Numerical experiment with up to non-dimensional wavenumber of 600 has successfully been conducted on a single laptop when one uses 100,000 degrees of freedom.

1. Introduction

Highly efficient simulation of high frequency sound wave propagation plays an important role in science and engineering fields [1,2], such as underwater sonar imaging detection [3] and active noise control analysis [4].

The propagation of sound wave in the isotropic medium is governed by a Helmholtz equation [5–7]. Unlike the Laplace problems [8–10], it is very difficult to solve efficiently the Helmholtz equation with large wavenumber due to the resulting coefficient matrix being large-scale and highly ill-conditioned.

On one hand, the high frequency methods have been widely applied for high frequency acoustic problems due to their less computational complexity and storage requirement using the geometric optics method (GO) [11], the geometrical theory of diffraction (GTD) [12] and the physical theory of diffraction (PTD) [13]. However, lower computational accuracy and instability restrict the applications of these methods for some acoustic problems having high precision requirements.

On the other hand, numerical simulation has merits of high accuracy. In recent years, a lot of new methods have been proposed for high frequency acoustic problems, such as the boundary knot method (BKM) [14–16], the asymptotic decomposition (AD) approach [17], the asymptotically derived boundary element method (ADBEM) [18] and the singular boundary method (SBM) [19–21]. They all have their own merits and drawbacks. The BKM can't be applied for exterior problems, and the obtained matrix is highly ill-conditioned. The AD approach and the ADBEM can only simulate the problems whose properties have been priori known. The SBM requires at least 6–8 boundary nodes in one wavelength per direction to produce an

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acceptable solution. To bypass these limitations, a modified singular boundary method (MSBM) was proposed recently [22]. Numerical investigation showed that the MSBM avoids the ill-conditioned matrix and needs only to place 2–3 source points in one wavelength per direction to produce the acceptable solution accuracy. However, the MSBM still has to be artificially truncated to satisfy the radiation boundary conditions at infinity for exterior problems.

The method of fundamental solutions (MFS) [23–25] is an efficient method to solve partial differential equations. The essential characteristic of the MFS is to introduce a fictitious boundary to solve the singularity of fundamental solutions. By the help of the fictitious boundary, the MFS can obtain highly accurate results using very few degrees of freedom (DOF) and CPU time. But the fictitious boundary also brings high uncertainty, which is the most important weakness of the MFS. Therefore, although the MFS has high numerical efficiency, it has rarely been applied for practical problems. Many researchers have proposed a lot of strategies [26,27] to remedy instability of the MFS. However, these strategies usually slow down the computation speed of the MFS and reduce its numerical efficiency. In addition, the physical reason why the fictitious boundary has such a dramatic effect on numerical results has never been explained successfully before.

In this study, we attempt to explain the reason why the fictitious boundary has such a dramatic effect on numerical results, and try to reveal how fictitious boundary affects numerical results. Based on this understanding, a dual-level method of fundamental solutions (DLMFS) with self-adaptive adjustment coefficients is proposed. The core feature of the DLMFS is to use a modified fundamental solution of the Helmholtz equation to replace the original fundamental solution. The method improves significantly stability, and it inherits high numerical efficiency of the MFS. The DLMFS places source points and their mirror source points on the fictitious boundary and physical boundary, respectively, and then combines the fundamental solutions generated by these two sets of source nodes as a modified fundamental solution of the Helmholtz equation. By introducing the concept of the equivalent slope, we provide an explicit empirical formula of the adjustment coefficient to eliminate the effect of different fictitious boundaries on numerical results.

Considering that the matrix obtained from the Helmholtz equation with high wavenumber usually has characteristics of large scale, high condition number (L2-norm) and high rank [28], solving of this type of matrix is a very difficult task. Therefore, we choose a high frequency acoustic problem to test the newly proposed DLMFS. In comparison with the existing methods [29–31], the DLMFS requires only to place 2 DOF in one wavelength per direction to produce acceptable solutions. This sampling frequency has already been down to the lowest one allowed by the sampling theorem. The other merit is that the DLMFS can automatically overcome non-uniqueness difficulty [32] for exterior acoustic problems. Thus the singular and hyper-singular integrals in the Burton–Miller formulations [33] are avoided. Subsequent numerical experiments carried out on a single laptop show that when one uses 100,000 DOF, the DLMFS can yield accurate solutions for exterior three-dimensional (3-D) pulsating sphere problems with up to non-dimensional wavenumber of $kd = 600$ ($d = 2\,\mathrm{m}$), where d is the maximum diameter of computational domain.

A brief outline of this article is as follows. Section 2 explains physical essence of the fictitious boundary and introduces the numerical methodology of the DLMFS. Section 3 investigates the DLMFS through two benchmark examples. Finally, some conclusions and outlooks are given in Sections 4 and 5.

2. Numerical methodology

2.1. Analysis of effect of the fictitious boundary on numerical results

In this section, we analyze how different fictitious boundaries affect numerical results in the MFS and explain the physical essence of the fictitious boundary, which constitute the main contribution of this study. For the convenience of physical deduction, the 3-D potential model is adopted as an example of illustration.

The whole solution process can be divided into three stages when one uses numerical methods to simulate a physical problem. They are the mathematical modeling, the numerical analyzing and the numerical solving. In the stage of mathematical modeling, a physical problem is reduced to partial differential equations. Model error arises from this stage. In the stage of numerical analyzing, one uses a numerical method to analyze the resulting partial differential equations, such as the BEM and the DLMFS. Discretization error arises from this stage. In the stage of numerical solving, one solves the obtained linear system of equations with the aid of a numerical solver, such as the Gauss solver and the generalized minimal residual algorithm solver. Truncation error arises from this stage.

The main error sources in the MFS are the discretization error and the truncation error. The proportion of weight of discretization error and truncation error decides the quality of numerical solutions. This proportion changes with location of the fictitious boundary. This is the reason why different fictitious boundaries have such a dramatic effect on numerical results.

At first, we analyze the relationship between discretization error and fictitious boundary. As is known to us all, numerical characteristics of the radial basis functions methods depend on property of the basis functions with change of distance. In this study, we use slope of the basis functions to illustrate its property related to distance. We suppose that p_0 is the average distance between the fictitious boundary and physical boundary. The function curve of fundamental solutions of the 3-D Laplace equation corresponding to p_0 is plotted in Fig. 1.

From the perspective of qualitative analysis, Fig. 1 can be divided into three parts according to slope of the fundamental solutions, namely, the sensitive part, the best part and the unresponsive part. When the distance between the fictitious boundary and the physical boundary is located on the sensitive part of the fundamental solutions, the fundamental solutions