Applied Mathematical Modelling 62 (2018) 21–37

Contents lists available at ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

An inequality unscented transformation for estimating the statistical moments

Zhang WenXin, Lu Zhenzhou*

Northwestern Polytechnical University, School Aeronaut, Xian 710072, Shaanxi, PR China

ARTICLE INFO

Article history:
Received 14 June 2017
Revised 13 April 2018
Accepted 14 May 2018
Available online 25 May 2018

Keywords:
Point estimate method
Unscented transformation
Uncertainty
Statistical moments
Nataf transformation

ABSTRACT

Point estimate method (PEM) is convenient for estimating statistical moments. This paper focuses on discussing the existing PEMs and presenting a new PEM for the efficient and accurate estimation of statistical moments. Firstly, a classification method of PEMs is proposed based on the strategy of choosing sigma points. Secondly, the minimum number of sigma points and the error of inverse Nataf transformation are derived corresponding to certain order and dimensionality of PEMs. Then the inequality unscented transformation (IUT) is presented to estimate the statistical moments. The proposed IUT permits the existing of limited errors in the matching of the first several order moments to decrease the number of sigma points, it opens new strategy of PEMs. The proposed method has two advantages. The first advantage is overcoming the growth of the number of sigma points with dimensionality since it parameterizes the number of sigma points and accuracy order. The second advantage is the wide applicability, for it has the ability to handle correlated and asymmetric random input variables and to match cross moments. Numerical and engineering results show the good accuracy and efficiency of the proposed IUT.

1. Introduction

Uncertainty widely exists in reliability evaluation, nonlinear filtering, probabilistic load flow calculation and other engineering problems, due to the inherent randomness of natural phenomena or the implicit and inaccurate assumptions related to the considered modeling approach [1,2].

There are always two ways to describe the uncertainty of output variables, probability density function (PDF) and statistical moments. PDF contains more information than statistical moments. But it is harder to estimate PDF than statistical moments. Sometimes information from statistical moments is enough for problems. If necessary, Edgeworth expansion and other approximations can be used to express PDF [3] once the statistical moments are obtained. So estimating statistical moments is a common method to solve uncertainty problem.

Several methods can be used for analyzing the statistical moments, such as Taylor series [4], Monte Carlo method (MCM) [5] or other sampling methods [6], response surface [7] or other surrogate approximation methods [8], point estimate method. Point estimate method (PEM) was first introduced by Rosenblueth [9] in 1975, and now there are many improvements [10–23]. Compared with other methods, PEMs have the following advantages [2,24,25]:

- The concept and the computational procedure are easy;

* Corresponding author.
E-mail addresses: qnq_zh@hotmail.com (Z. WenXin), zhenzhoulu@nwpu.edu.cn (L. Zhenzhou).

https://doi.org/10.1016/j.apm.2018.05.010

- It does not need to calculate the gradients of output function;
- It only needs the first few statistical moments of the input variables.

PEMs have been successfully applied in different fields. In reliability analysis, inputs and outputs can be regarded as random variables. The statistical moments of the output are used to calculate the failure probability and the reliability sensitivity [23]. So PEMs can be used in importance measure [26–28] and the reliability analysis [29–32]. In filtering, unscented Kalman filter (UKF) is a widely used method for nonlinear systems [33]. UKF provides an efficient computational method to estimate the state of a process, so it is used in control systems [34]. PEMs are also used in probabilistic load flow (PLF). PLF considers volatility of load, random outage changes of generator [35] and other uncertain factors. So PEMs can be used in PLF to get the electric parameters of the power networks [1,36–38].

A new and efficient PEM is developed in this paper. This method can control the high growth problem of the number of sigma points (discussed in Section 4) in high dimensionality problems of PEMs. In order to show the advantages of proposed method, several discussions of PEMs are necessary. Firstly, most existing PEMs are classified into five categories. Secondly, whether Nataf transformation is suitable for PEMs is discussed. Moreover, a lower bound of the number of sigma points with certain order accuracy and dimensionality is derived.

The rest of the paper is organized as follows: Section 2 reviews the PEMs. After a new classification method of PEMs proposed in Section 2.1, several representative PEMs are analyzed about their principle, advantages and disadvantages. In Section 3, two shortcomings of existing PEMs are analyzed based on the mathematical essence of PEMs, and the error of Nataf transformation used by the existing PEMs is devised theoretically. Section 4 proposes a new PEMs called inequality unscented transformation (IUT). Section 5 employs examples to show the accuracy and efficiency of the presented method compared with other PEMs. Section 6 gives the conclusions and works in progress.

2. Review of PEMs

Point estimate method (PEM) was first introduced by Rosenblueth [9] in 1975. Since then, several improved methods, such as Harr's method [10], Hong's method [11], were presented. Besides, in 1995, Julier et al. [12] proposed unscented transformation (UT), which is a part of unscented Kalman filter in nonlinear filtering. As an efficient PEM, UT has been improved by scaled unscented transformation (SUT) [13], high order unscented transformation (HOUT) [14,15], conjugate unscented transformation (CUT) [16], etc. Some other filtering methods, such as cubature Kalman filter (CKF) [20] and Gauss–Hermite Kalman filter (GHKF) [18], also use their own principles to estimate the statistical moments.

Consider a function $Y = f(\boldsymbol{X})$ involving n-dimensional random input vector $\boldsymbol{X}$ with mean $\bar{\boldsymbol{X}}$, variance $[\sigma_1^2, \dots, \sigma_n^2]$ and covariance matrix $\boldsymbol{R}$. Estimate points $(\boldsymbol{S_i} + \bar{\boldsymbol{X}})$ and weights w_i can be calculated by a certain PEM. Then the statistical moments of Y can be estimated by l sigma points.

$$E(Y) \approx \sum_{i=1}^{l} w_i f\left(\boldsymbol{S}_i + \bar{\boldsymbol{X}}\right) \tag{1}$$

$$E[Y - E(Y)]^k \approx \sum_{i=1}^{l} w_i \left[f\left(\boldsymbol{S}_i + \bar{\boldsymbol{X}}\right) - \sum_{j=1}^{l} w_j f\left(\boldsymbol{S}_j + \bar{\boldsymbol{X}}\right) \right]^k \tag{2}$$

Eq. (1) is the estimated mean of Y, while Eq. (2) is the estimated kth order central moments of Y. The different PEMs have their own algorithms searching for sigma points and weights. Zhao & Ono's method and Zhou & Nowak's method have different formulas for estimating central moments [19,23]. But these methods can be reformed as Eqs. (1) and (2).

The basic idea of PEMs is using a weighted sum of the function f in some deliberately chosen points $\boldsymbol{S}_i$ to estimate the statistical moments of the output variable Y. The deliberately chosen points are called sigma points in filtering, and these chosen points are called as the sigma points for all PEMs in this paper. Different PEMs differ in the strategy of choosing sigma points and weights [39]. In Section 2.1, a classification method of PEMs is proposed based on the strategy of choosing sigma points in input variable space.

2.1. A new classification method of PEMs

There are about twenty different PEMs available now. Some PEMs have similar strategies in choosing sigma points. Many literature proposed classifications of the PEM. Fan et al. [40] classified PEMs according to that the sigma points were produced in the original variable space or in the reference variable space. Zhang et al. [41] classified PEMs according to the fixed or unfixed of accuracy order and number of sigma points. Christian and Baecher [42] used Hypercube PEM and Hypersphere PEM to distinguish different PEMs. In this section, a new classification method of PEMs is proposed. The PEMs are divided into five categories, which are Hypercube PEM, Hypersphere PEM, Multi-Layer Hypersphere PEM, High Order PEM and Precision-Variant PEM.

The first category is *Hypercube PEM*. Hypercube PEM includes Rosenblueth's method, 3-point estimate [17], the PEM of Gauss–Hermite Kalman filter [18], etc. The configuration of sigma points of Hypercube PEM in n-dimensional symmetrical standardized distributions space is a Hypercube. Fig. 1 is the Rosenblueth's method for 3-dimensional independent standardized normal distribution input variables. Hypercube PEMs has the problem of the exponential-increasing number of