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Nonstandard finite difference scheme for a Tacoma Narrows Bridge model

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ABSTRACT

This paper constructs two dynamically consistent nonstandard finite difference (NSFD) schemes for the model of Tacoma Narrows Bridge using the Mickens methodology. The model consists of nonlinear, coupled, second order ordinary differential equations (ODEs). The standard forward Euler fails to capture the correct behavior of the system for a given step size. However, using the same step size, both NSFD schemes correctly approximate the solution to the system.

1. Introduction

Construction began on the Tacoma Narrows Bridge in Tacoma, Washington on September 27, 1938. During construction, unusual rhythmic vertical motion of the bridge was observed [1]. The bridge was completed and opened to the public on July 1, 1940. This gave the public the opportunity to experience the motion firsthand. On November 7, 1940, the Tacoma Narrows Bridge collapsed into the Puget Sound. The reason for its collapse has been a widely debated topic [2–5]. Various mathematical models have been put forth in an attempt to explain the behavior of the bridge that led to its collapse [3,6,7]. Based on eyewitness accounts, the bridge did not only display vertical oscillations, but torsional oscillations as well [7].

We will focus on constructing nonstandard finite difference schemes to approximate solutions to the models put forth by McKenna and co-workers [5,7–9]. McKenna's equation for the coupled torsional and vertical oscillation of the bridge is a nonlinear differential equation. At the time, no technology was available to solve this nonlinear equation [7]. This limitation caused engineers who were studying possible motions of the bridge to linearize the equation in order to solve it [7]. However, this linearized model failed to predict the possible high amplitude torsional behavior of the bridge. In [7], McKenna emphasized the need to numerically investigate the nonlinear equations as they more accurately capture the phenomena that the bridge exhibited on the day of its collapse. Since no closed solution exists for the Tacoma Narrows Bridge models, we seek to discretize the systems.

Standard numerical techniques have been utilized for a long time. For example the forward Euler scheme which was first published in the 1960s [10]. Nonstandard finite difference (NSFD) schemes on the other hand were not introduced until the 1980s, with Potts' interest in finding better approximate solutions to nonlinear differential equations [11]. Mickens then expanded on the idea of NSFD schemes by constructing such schemes that are dynamically consistent with a given system [12]. For example, when standard numerical integration methods, such as the forward Euler scheme, are used, numerical instabilities may occur. A finite difference scheme is said to have numerical instabilities if there are solutions to the finite

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difference scheme that do not qualitatively match solutions to the corresponding differential equation. Take for example the special case of the decay equation with $\lambda = 1$

$$\frac{dy}{dt} = -y \tag{1}$$

and initial condition

$$y(t_0) = y_0. \tag{2}$$

By separation of variables, the exact general solution for Eq. (1) can be derived and is given by

$$y(t) = y_0 e^{-(t-t_0)}. \tag{3}$$

Note that $y(t)$ in (3) decreases monotonically to zero as t increases without bound.

The forward Euler scheme for Eq. (1) is given by

$$\frac{y_{k+1} - y_k}{h} = -y_k \Rightarrow y_{k+1} = (1-h)y_k, \tag{4}$$

and has general solution

$$y_k = y_0(1-h)^k. \tag{5}$$

The forward Euler scheme (4) when used on the decay equation (1) will experience numerical instabilities when the value of the step size h falls outside the interval $0 < h < 1$.

Numerical instabilities such as the one shown above may be avoided by using NSFD schemes. An NSFD scheme is a difference scheme which seeks to avoid numerical instabilities by incorporating dynamics of the system into the scheme. Dynamical consistency is achieved when the difference equation has the same qualitative properties as its corresponding differential equation. To create a suitable NSFD scheme for a differential equation, a set of modeling strategies can be used to optimize dynamical consistency [13]. Some of these strategies are as follows:

1. Exact finite-difference schemes generally require that nonlinear terms be modeled nonlocally [14].
2. The discrete derivatives for differential equations have denominator functions $\varphi(h)$ that are typically more complicated than those used in the standard modeling procedure. For example, the denominator function $\varphi(h)$ of a first order differential equation has the property $\varphi(h) = h + \mathcal{O}(h^2)$. This allows for the construction of a larger class of finite difference models and also provides for more ambiguity in the modeling process [13,15].
3. The order of the discrete derivatives in exact finite difference schemes is always equal to the corresponding order of the derivatives of the differential equation.
4. Important properties of the differential equations and/or their solutions should be incorporated into their corresponding discrete forms. Properties such as positivity of solution, traveling wave solutions, periodic solutions, etc., should be preserved, if possible.

These modeling strategies have become the basis for creating NSFD schemes.

For example, for the decay equation (1), by using the following substitutions,

$$\begin{cases} t \to t_{k+1} \\ t_0 \to t_k \\ y(t) \to y_{k+1} \\ y(t_0) \to y_k \end{cases} \tag{6}$$

where $h > 0$ and $k = 0, 1, 2, 3, \ldots,$ an exact NSFD scheme is given by

$$\frac{y_{k+1} - y_k}{1 - e^{-h}} = -y_k. \tag{7}$$

This scheme is exact because solving for the $(k+1)$th time-step,

$$y_{k+1} = y_k e^{-h} = y_0 e^{-(k+1)h} = y_0 e^{-(t_{k+1} - t_0)} \tag{8}$$

which equals the analytical solution for all time $t = t_k$. Thus, regardless of the step size h, the NSFD scheme (7) does not experience any numerical instabilities.

The main purpose of this paper is to design an NSFD scheme to numerically solve a nonlinear model of the Tacoma Narrows Bridge. There are a myriad ways to handle discretizing differential equations, each with their own advantages. Creating an NSFD scheme for this model gives us a different way of approaching the discretization process that may better preserve the solution dynamics since the system is taken into account in the construction of the scheme. In general, when faced with a differential equation with no closed solution, we have an additional technique, namely, the NSFD method, that may perform better than standard techniques.

In this paper, we compare the performance of the standard and nonstandard schemes to the performance of MATLAB's `ode45` solver [16,17] since a closed solution does not exist. Note that the solutions obtained by using `ode45` are not assumed to be the only way to discretize the system. The important aspect of this paper is that our discretization have the