Applied Mathematical Modelling 62 (2018) 664–679

Contents lists available at ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

A new point estimation method for statistical moments based on dimension-reduction method and direct numerical integration

Wenliang Fan[a,b,*], Runyu Liu[a], Alfredo H-S Ang[c], Zhengliang Li[a,b]

[a] School of Civil Engineering, Chongqing University, Chongqing 400045, China
[b] Key Laboratory of New Technology for Construction of Cities in Mountain Area (Chongqing University), Ministry of Education, Chongqing 400045, China
[c] Department of Civil and Environmental Engineering, University of California, Irvine, CA 92697, USA

ARTICLE INFO

Article history:
Received 2 July 2017
Revised 11 May 2018
Accepted 11 June 2018
Available online 24 June 2018

Keywords:
Statistical moments
Point estimate method
Dimension-reduction method
High dimensional integration
Curse of dimension

ABSTRACT

Estimation of statistical moments of structural response is one of the main topics for analysis of random systems. The balance between accuracy and efficiency remains a challenge. After investigating of the existing point estimation method (PEM), a new point estimate method based on the dimension-reduction method (DRM) is presented. By introducing transformations, a system with general variables is transformed into the one with independent variables. Then, the existing PEMs based on the DRMs are investigated. Based on the qualitative analysis of difference in the approximations for response function and moment function, a new PEM is proposed, in which the response function is decomposed directly and the moments are calculated by high dimensional integral directly. Compared with the existing PEM based on univariate DRM, the proposed method is more friendly and easier to implement without loss of accuracy and efficiency; as compared with the PEM based on the generalized DRM, the proposed method is of better precision at the cost of nearly the same efficiency and computational complexity, further, it does hold that the even-order moments are nonnegative. Finally, several examples are investigated to verify the performance of the new method.

1. Introduction

For time-invariant stochastic systems, stochastic response is a random variable and can be described exactly by either the probability density function (PDF) or infinite statistical moments. In general, the first several moments are useful and accurate sufficiently for probabilistic information of the random variable, and are much easier to calculate than the PDF. Therefore, evaluating the first several moments of structural response becomes one of the main topics of random system analysis.

Obviously, stochastic response Z is the function of the random vector of system, i.e. $\boldsymbol{\Theta} = (\Theta_1, \Theta_2, \ldots, \Theta_n)$, in which n is the number of the random variables, and can be formulated as

$$Z = g(\boldsymbol{\Theta}) = g(\Theta_1, \Theta_2, \ldots, \Theta_n) \tag{1}$$

* Corresponding author at: School of Civil Engineering, Chongqing University, Chongqing 400045, China.
E-mail addresses: davidfwl@126.com, davidfwl@cqu.edu.cn (W. Fan).

https://doi.org/10.1016/j.apm.2018.06.022

Clearly, the mean value, μ_Z, and the qth central moments, $m_{Z,q}$, of Z are expressed as follows:

$$\mu_Z = \int_{\Omega_{\boldsymbol{\Theta}}} g(\boldsymbol{\theta}) p_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) \mathrm{d}\boldsymbol{\theta} \approx \sum_{l_1=1}^{r_1} \sum_{l_2=1}^{r_2} \dots \sum_{l_n=1}^{r_n} w_{\theta,l_1,l_2,\dots,l_n} g(\theta_{1,l_1}, \theta_{2,l_2}, \dots, \theta_{n,l_n}) \tag{2}$$

$$m_{Z,q} = E\left[(g(\boldsymbol{\Theta}) - \mu_Z)^q\right] = \int_{\Omega_{\boldsymbol{\Theta}}} (g(\boldsymbol{\theta}) - \mu_Z)^q p_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) \mathrm{d}\boldsymbol{\theta} \approx \sum_{l_1=1}^{r_1} \sum_{l_2=1}^{r_2} \dots \sum_{l_n=1}^{r_n} w_{\theta,l_1,l_2,\dots,l_n} \left[g(\theta_{1,l_1}, \theta_{2,l_2}, \dots, \theta_{n,l_n}) - \mu_Z\right]^q \tag{3}$$

in which $\Omega_{\boldsymbol{\Theta}}$ is the distribution domain of $\boldsymbol{\Theta}$, and $p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$ is the joint PDF of $\boldsymbol{\Theta}$, $(\theta_{1,l_1}, \theta_{2,l_2}, \dots, \theta_{n,l_n})$ and $w_{\theta,l_1,l_2,\dots,l_n}$ are the abscissas and weights of a quadrature rule with the weight function $p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$, respectively, r_i is the number of the abscissas for Θ_i. $E(\cdot)$ denotes the expectation. Especially, if the components of $\boldsymbol{\Theta}$ are mutually independent, there exists

$$p_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) = \prod_{i=1}^{n} p_{\Theta_i}(\theta_i) \tag{4}$$

$$w_{\theta,l_1,l_2,\dots,l_n} = \prod_{i=1}^{n} w_{\theta_i,l_i} \tag{5}$$

where $p_{\Theta_i}(\theta_i)$ is the marginal PDF of Θ_i, w_{θ_i,l_i} is the weight corresponding to $p_{\Theta_i}(\theta_i)$.

It is obvious from Eqs. (2) and (3) that $N_{\text{total}} = \prod_{i=1}^{n} r_i$ structural re-analyses are required for μ_Z and $m_{Z,q}$. In general, $r_1 = \dots = r_n = r$, and $N_{\text{total}} = r^n$, which increases exponentially with n, and means the curse of dimension on structural analysis. Further, the number of algebraic operation, such as multiplication and additive operation, is of the same order of N_{total} in Eqs. (2) and (3). In other words, the curse of dimension on algebraic operation also exists for moment evaluation. Comparatively speaking, structural analysis is much more complex and time-consuming than algebraic operation. Hence, it draws much more attention on alleviating the curse of dimension of structural analysis.

The point estimate method (PEM) is an efficient and easy-to-implement method for moment evaluation that can alleviate the curse of dimension on structural analysis to some extent, in which there are mainly two strategies for a multivariable function.

The first strategy refers to as the point-selection strategy. Rosenblueth [1,2] set $r = 2$ and then $N_{\text{total}} = 2^n$, which is computationally acceptable as n is not very large. By introducing the eigen system of the correlation matrix in a hypersphere, Harr [3] and Chang et al. [4,5] proposed PEMs with only $2n$ structural analyses, which improves the efficiency of the PEM tremendously. However, the accuracy of the PEMs above is not sufficient for the higher statistical moments of most of structural system and even the lower moments for the strongly nonlinear systems. Hong [6] took into account higher-order moments of random variables, and can calculate higher-order moments of response with only $N_{\text{total}} = rn$. However, the points are not easy to obtain and the precision may be not sufficient for system with much more cooperative effects among variables.

The second strategy is approximating the response function by a series of lower-dimension component functions, namely the dimension decomposing (DD), including dimension-reduction method (DRM) [1,7–16] and high dimensional model represent (HDMR) [17–24]. According to the difference on format of component functions, there are two kinds of DDs. In the first kind, the component functions have similar format with the original response function and are usually implicit for practical structure [7–16], which can be named as the direct DD; for the second kind, the component functions in the direct DD are further approximated by explicit polynomials, such as orthogonal polynomials [17,18] and interpolation polynomials (including Lagrange interpolation polynomials [19–22], moving least squares interpolation polynomials [23,24], and so on). Since explicit approximations of component functions are obtained, the second kind of DD is naturally applied in reliability [19,20,23,24], sensitivity [21,22], optimization [18] and so on. However, for moment evaluation, the second kind of DD is not very convenient, and the reason is that: (1) explicit approximations are unnecessary for moment estimation; (2) in the explicit approximation, there are many coefficients, which are complicated to calculate. Therefore, the first kind of DD is used more frequently than the second one in moment estimation of response. Because DD is referred to DRM in most of researches on DD based moment estimation [1,7–16], the direct DD is also referred to DRM hereafter in this work.

In the DRM, even though there are two types of approximations of multivariable functions, namely product approximation [1,7] and summation approximation [8–16], the latter one attracts much more research interesting. In Ref. [9], the response function was approximated by a sum of univariate polynomials and the 2nd order cross terms of all variables, in which the partial derivatives are calculated by the finite difference method, $r = 3$ and $N_{\text{total}} = (n^2 + 3n + 2)/2$. Because $r = 3$, this method is not suitable for higher moments; further, the length of the difference step will affect the precision of the method. Both Zhao & Ono [10,11] and Rahman and Xu [12] proposed the summation approximation by a series of one-dimension component functions, in which $N_{\text{total}} = n \times r + 1$ at most for both methods. The generalized s-DRM [13] becomes a promising and usually more accurate method, in which s-dimensional interactive effects among variables are involved, and $N_{\text{total}} = \sum_{i=0}^{s} \binom{n}{i} r^i$ at most. Obviously, the efficiency of the generalized DRM decreases significantly with s, but the precision is usually acceptable for small s, such as 2 or 3, in practice.

For a given multivariate function, the approximation based on the generalized DRM is usually more accurate than the one based on the univariate DRM as $s > 1$, since more real existing interactive effects among variables are taken into consideration. Unfortunately, the existing PEM based on the univariate DRM approximates the original response function directly,