Applied Mathematical Modelling 60 (2018) 1–17

Contents lists available at ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

A general high order two-dimensional panel method

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ARTICLE INFO

Article history:
Received 11 May 2017
Revised 22 December 2017
Accepted 13 February 2018
Available online 21 February 2018

Keywords:
Boundary element method
Vortex method
Airfoil design
Potential flow
Fast multipole method

ABSTRACT

We develop an efficient and high order panel method with applications in airfoil design. Through the use of analytic work and careful considerations near singularities our approach is quadrature-free. The resulting method is examined with respect to accuracy and efficiency and we discuss the different trade-offs in approximation order and computational complexity. A reference implementation within a package for a two-dimensional fast multipole method is distributed freely.

1. Introduction

Most methods to numerically solve partial differential equations (PDEs) fall into one of two categories. The first is volume discretization methods, including, for example, finite element and finite volume methods. Here the resulting set of equations is large but sparse since the discretization nodes are connected only locally. If the PDE has a known fundamental solution, the full solution may instead be obtained in a discretization procedure involving only the boundary. This approach is commonly known as boundary element methods (BEMs) and generally involves fewer unknowns which, however, are connected globally.

In the current work, focus will be on fluid mechanical applications where the Laplace equation is used to calculate potential flow solutions. For many aerodynamics simulations, the target is a small object in the form of, e.g., an airfoil in a large domain which makes the BEM particularly attractive [1–3]. For time-dependent calculations, the method is commonly combined with the release and subsequent advection of *vortices*, effectively discretization points which approximates the flow. This is the method known as the *vortex method* [4]. Here the flow velocity $\vec{V}$ is calculated as a combination of a potential flow and of vortex contributions,

$$\vec{V} = \nabla\phi + \vec{V}_{\omega}, \tag{1.1}$$

where ϕ is the solution to a Laplace equation and where $\vec{V}_{\omega}$ is the contribution from the vortices in the flow [3,5,6]. Note that the contribution to the flow velocity from each vortex has to be calculated at each vortex position and at every time step. This is an *N-body problem* for which, as discussed below, fast algorithms should be employed.

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https://doi.org/10.1016/j.apm.2018.02.010

One common method for solving the potential flow problem is to discretize the boundaries using *panels*. For fluid mechanical applications, these panels are constructed such that the flow satisfy the no-penetration boundary condition,

$$\vec{V} \cdot \hat{n} = 0. \tag{1.2}$$

The standard approach in 2D is to discretize the boundary using linear panels consisting of straight line segments, and to use a panel strength which is either constant or linearly varying along this line. These panels can be constructed from source and vortex sheets, but other possibilities also exist [7].

An obvious issue with panels that have a linear shape is that there will be sharp corners at the transitions between panels such that the solution to Laplace's equation approaches infinity at the corner. For a pure potential flow solution, this is often not a problem, since the no penetration boundary condition is only satisfied at the centers of the panels. For vortex methods, however, this yields large numerical errors when evaluating the velocity in the vicinity of such transition points. A remedy is to use panels with higher order shapes to make the boundary smooth. For three-dimensional implementations, such panels have been discussed in [8–10]. Here, numerical integration is required to solve the flow contribution from the panel. This can be time consuming, especially for velocity evaluations close to the singularity of the fundamental solution, and it is therefore desirable to avoid numerical integration whenever possible. Indeed, for two-dimensional calculations, Ramachandran and co-authors have derived a solution involving panels with cubic shape and a linear distribution of the panel strength [11].

To improve the computational speed, vortex methods commonly rely on the fast multipole method (FMM) [12], and the same method can also be used to accelerate the solution of the dense BEM matrix with the influence coefficients of the panels [11,13].

In the present paper we extend the work in [11] and design a general framework for two-dimensional panels with high order in both shape and strength and which does not require numerical integration. We develop the necessary analytic relations in Section 2, where we also discuss practical implementation issues allowing the method to be evaluated via the FMM. The performance of our framework is evaluated in Section 3 and conclusions are summarized in Section 4.

Our method has been implemented within the 2D FMM-software described in [14,15], and is distributed as open source. See Section 4.1 for details.

2. Theory and implementation

We construct the panels from point sources/vortices in Section 2.1. The procedure for calculating the contribution from a panel is developed in Section 2.2, including both near and far-field evaluations. How the boundary conditions can be solved is described in Section 2.3, which includes how to integrate the contribution of a source point/panel over the panel surface. Finally, corrections to ensure continuity of the source strength are discussed in Section 2.4

2.1. Representation of panels

For a two dimensional flow using the complex number representation, the velocity V at position z from a source/vortex at position z_v is given by

$$\overline{V(z)} = \frac{1}{2\pi} \frac{Q + i\Gamma}{z - z_v}, \tag{2.1}$$

where $\overline{V(z)}$ denotes the complex conjugate of the velocity, Q is the source strength, and Γ is the vortex strength.

Assume that a part of the boundary of an object extends between the points z_1 and z_2. To generate a high order panel that models this boundary, we need to select a baseline along which the panel is parameterized. Since only the direction of this reference line will be of importance, we let it start at z_1 with some angle θ pointing in the general direction of z_2, see Fig. 2.1. The natural choice is to choose the baseline between z_1 and z_2, but to allow for the panel to be split (as relied upon in the FMM), it is necessary to allow that z_2 is not on the baseline.

Following the notation of Ramachandran [11], if we are interested in the flow velocity at position z, we apply the transformation

$$z' = z'(z) = (z - z_1)e^{-i\theta}, \tag{2.2}$$

which will make the reference line parallel to the real axis and the panel will extend between 0 and λ, where

$$\lambda = \mathrm{Re}\left\{(z_2 - z_1)e^{-i\theta}\right\}. \tag{2.3}$$

Using this reference system we can write a position z_p on the panel as

$$z_p = z_p(\zeta) = \zeta + i\eta(\zeta) = \zeta + i\sum_{k=0}^{M} a_k \zeta^k, \tag{2.4}$$

for a panel shape defined by a polynomial of degree M. To ensure that the shape of the panel remains numerically reasonable, it will be assumed that the coefficients a_k are real.