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Helical flow of a Bingham fluid arising from axial Poiseuille flow between coaxial cylinders

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ABSTRACT

The Bingham fluid model represents viscoplastic materials that display *yielding*, that is, behave as a solid body at low stresses, but flow as a Newtonian fluid at high stresses. In any Bingham flow, there may be regions of solid material separated from regions of Newtonian flow by so-called *yield boundaries*. Such materials arise in a range of industrial applications. Here, we consider the helical flow of a Bingham fluid between infinitely long coaxial cylinders, where the flow arises from the imposition of a steady rotation of the inner cylinder (annular Coutte flow) on a steady axial pressure driven flow (Poiseuille flow), where the ratio of the rotational flow compared to the axial flow is small. We apply a perturbation procedure to obtain approximate analytic expressions for the fluid velocity field and such related quantities as the stress and viscosity profiles in the flow. In particular, we examine the location of yield boundaries in the flow and how these vary with the rotation speed of the inner cylinder and other flow parameters. These analytic results are shown to agree very well with the results of numerical computations.

1. Introduction

The Bingham fluid (Bingham [1]) has the property that it behaves as a solid until the internal fluid stress reaches a critical value, termed the yield stress, and then, for stress states beyond the yield value, flows as a viscous Newtonian fluid, with stress proportional to the rate of shearing. Thus, it belongs to the class of non-Newtonian yield stress fluids. In flows of such fluids, stress variation may give rise to fluid and solid zones in the flow field, separated by yield boundaries, at which the stress takes on the yield value. This feature of the Bingham fluid model has been exploited in modeling the flow of a wide range of yield stress fluids, ranging from tomato sauce (ketchup) [2], wet concrete [3] and cake formation in mud between the bore hole and the drill stringer [4].

Here, we consider flow of a Bingham fluid in the gap between infinitely long circular coaxial cylinders. This flow consists of two components - an axial Poiseuille flow, driven by a constant axial pressure gradient, superimposed on a transverse rotational Couette flow, arising from the steady rotation of the inner cylinder about its axis. Apart from this rotation, all bounding cylinder surfaces are stationary. This combination generates a helical fluid flow, in which any moving fluid particles trace out helical paths with axes coincident with the common axis of the cylinders.

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Such helical flows are of interest in applications, particularly in modeling the action of an on-line flow rheometer (see Bown et al. [5] or Nguyen et al. [6]). For the helical flow of Newtonian fluids, the equations for the fluid field are readily integrated to give an exact solution (see Langlois [7]). However, for non-Newtonian fluids (yield stress or otherwise), the nonlinearity of the fluid field equations makes this task considerably more difficult and usually numerical solution methods must be used.

The helical flow problem has been solved by Coleman and Noll [8], with the components of the fluid velocity field and associated constants of integration being represented in terms of integrals. Evaluation of these for any Non-Newtonian fluid would almost certainly require a numerical process. A similar general approach (with comparable results) was adopted by Fredrickson [9]. For yield stress fluids (and the Bingham fluid in particular), the possibility of fluid and solid regions separated by yield boundaries makes solving the problem even more complex.

A considerable simplification occurs when the inner cylinder is stationary, so that the Bingham fluid flows in a steady axial Poiseuille flow driven by an axial pressure gradient. This situation was analysed by Fredrickson and Bird [10] in the case where the inner and outer cylinders were stationary. They demonstrated the existence of a flow field consisting of a solid moving core (plug zone) in the intercylindrical gap, separating fluid regions adjacent to the inner and outer cylinder walls.

More recently, this situation has been extended by Liu and Zhu [11], to the case of a Bingham fluid in axial Couette–Poiseuille flow, where the inner and/or outer cylinder(s) are in steady axial motion. Building on earlier work by Filip and David [12] for power law fluids, they demonstrated the possibility of a detached plug zone (as in [10]), but also the possibility of a plug zone attached to the inner or outer cylinder. They also found flow regimes in which no plug zones occurred at all, with the fluid in pure shear flow in the intercylindrical gap.

In both [10] and [2], the flow was not helical, but rectilinear along the common axial direction of the bounding cylinders. An extension to true helical flow was carried out by Rao [13], for the case where the inner cylinder was fixed and the outer cylinder was in constant rotational and axial translational motion. There was zero axial pressure gradient, so both fluid flow components(rotational and translational) were Couette in nature. A mixture of analytical and numerical techniques were used to study the case where a solid plug is attached to the outer cylinder and there is a fluid zone adjacent to the inner cylinder, and when there is plastic flow throughout the cylinder.

More recently, Bittleston and Hassager [14] have examined the helical flow we consider here, with inner cylinder rotating with constant angular velocity, outer cylinder motionless and with an axial pressure gradient. By regarding the cylindrical flow region to be narrow, they approximate the flow by that of a Bingham fluid between plane regions and derive an approximation to the flow field. They also obtain conditions governing the existence (or not) of a floating core in the intercylindrical gap, plus approximations to the core width and velocity. Limits of the validity of this approach are investigated numerically and extensions of the method to other yield stress fluids are discussed.

As noted above, here we consider helical flow, consisting of the pressure driven flow situation of Fredrickson and Bird [10], with an established floating core in the intercylindrical gap, upon which we superimpose the effect of the inner cylinder rotating with a constant angular speed. This scenario has sufficient complexity to rule out any likelihood of (exact) analytical solutions to the equations of motion and in general necessitates the use of numerical solution techniques. However, our analysis will identify a dimensionless parameter, m, that is small in physically realistic situations, with $m = 0$ corresponding to the pure axial flow of [10]. The small parameter m is essentially the ratio of the torque per unit length, used to rotate the inner cylinder, to the axial pressure gradient, producing the primary axial flow.

Thus, in the following sections, we analyse the helical flow of a Bingham fluid as described above, using m as a perturbation parameter. We will use this analysis to obtain approximate analytical expressions for the fluid velocity, the structure of the flow field and the variation of the fluid stress and viscosity, that are valid over the whole flow domain. These expressions will be valid for a range of values of the parameters occurring in the flow problem, in contrast with the results of numerical computations, valid only for specific parameter values.

We will compare these approximations with the results of numerical computations for particular parameter values as a verification. Such an approach has already been used in the analysis of helical flows of non-Newtonian fluids related to the rheometer application referred to above (Bhattacharya et al. [15], Shepherd et al. [16] or Farrugia et al. [17]).

It is important to note that of the earlier works described above ([12]–[14]), only the last two consider a truly helical flow; and only the last, [14], incorporates pressure driven (Poiseuille) flow. Moreover, our perturbation analysis gives approximate results that apply over the whole of the intercylindrical gap, not a localized region, as in [14]. Thus, it should be seen as a distinct and original contribution to this flow problem.

We note that this approach may be compared to that of a recent paper by Shepherd, Stacey et al. [18], which analyzed a Bingham flow which comprised an annular Couette flow with a core attached to the outer (stationary) cylinder upon which was superimposed an axial flow of given axial flow rate. There, the identified small parameter was related to the axial flow rate; and indeed is related to the reciprocal of the parameter m described above. See Alharbi [19] for further discussion of this. Thus, direct comparisons of the results of [18] and the results to be obtained here are not possible. The two flow problems are distinct.