Applied Mathematical Modelling 56 (2018) 15–31

Contents lists available at ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

Symbolic and numeric scheme for solution of linear integro-differential equations with random parameter uncertainties and Gaussian stochastic process input

I.E. Poloskov[a,*], C. Soize[b]

[a] Faculty of Mechanics and Mathematics, Perm State University, 15 Bukirev street, Perm, GSP, 614 990, Russia
[b] Laboratoire Modélisation et Simulation Multi Echelle, MSME UMR 8208 CNRS, 5 bd Descartes, Marne-la-Valle, 77454, France

A R T I C L E I N F O

Article history:
Received 18 May 2017
Revised 28 September 2017
Accepted 20 November 2017
Available online 2 December 2017

Keywords:
Stochastic integro-differential equation
Random parameter
Gaussian input
Green function
Second-order moments' calculation
Monte Carlo simulation

A B S T R A C T

The paper describes a theoretical apparatus and an algorithmic part of application of the Green matrix-valued functions for time-domain analysis of systems of linear stochastic integro-differential equations. It is suggested that these systems are subjected to Gaussian nonstationary stochastic noises in the presence of model parameter uncertainties that are described in the framework of the probability theory. If the uncertain model parameter is fixed to a given value, then a time-history of the system will be fully represented by a second-order Gaussian vector stochastic process whose properties are completely defined by its conditional vector-valued mean function and matrix-valued covariance function. The scheme that is proposed is constituted of a combination of two subschemes. The first one explicitly defines closed relations for symbolic and numeric computations of the conditional mean and covariance functions, and the second one calculates unconditional characteristics by the Monte Carlo method. A full scheme realized on the base of *Wolfram Mathematica* and *Intel Fortran* software programs, is demonstrated by an example devoted to an estimation of a nonstationary stochastic response of a mechanical system with a thermoviscoelastic component. Results obtained by using the proposed scheme are compared with a reference solution constructed by using a direct Monte Carlo simulation.

1. Introduction

1.1. Subject area, models, review of tools, and structure of the paper

Deterministic ordinary integral differential equations (OIDE) and stochastic ordinary integral differential equations (SOIDE) are interesting from both academic scientific and technical points of view because these equations are models of phenomena in a huge number of different sectors. A common theory and primary classification of deterministic partial integral differential equations (PIDE) was developed by Vito Volterra in the first half of twentieth century. General ideas of stochastic dynamics have been considered, for instance, in [1–6].

In many cases, sources of models in the form of SOIDE in stochastic mechanics are results of space discretizations of stochastic partial integro-differential equations (SPIDE) that describe continuous viscoelastic media [7]. Techniques of such

* Corresponding author.
E-mail addresses: igor.poloskov@psu.ru (I.E. Poloskov), christian.soize@univ-paris-est.fr (C. Soize).

https://doi.org/10.1016/j.apm.2017.11.024

discretizations are usually based on well-known ideas of the finite element method, the finite difference method and another computation schemes for solution of space-time problems.

The first investigations of viscoelastic behavior were made by W.E. Weber, R. and F. Kohlrausch, L. Boltzmann, O. Meyer, D.K. Maxwell, W. Thomson (Lord Kelvin), W. Voigt, P. Duhem, L. Natanson, S. Zaremba, A.E. Green and R.S. Rivlin (1957), B.D. Coleman and W. Noll (1958, 1961, 1964), mainly in the second half of nineteenth century. Historical remarks about developments in the viscoelastic domain can be found in [8].

Foundations of modern theory of viscoelasticity were expounded in [8]. An important part of this theory is an apparatus for investigation of time-nonhomogeneous systems [9]. In the case of linear aging viscoelasticity, where the term *aging* means that the mechanical properties of a given material are changed with its age, the constitutive equations for linear viscoelastic media with infinitesimal strains can be expressed as Stieltjes integrals (according to Riesz's representation theorem [8]), and are written as

$$\sigma_{ij}(r,t) = \sum_{k,\ell=1}^{3} \int_{t_0}^{t} F_{ijk\ell}(r,t,\tau)\, d\varepsilon_{k\ell}(r,\tau), \quad \sigma_{ij}(r,t_0) = 0, \tag{1}$$

$$\varepsilon_{ij}(r,t) = \sum_{k,\ell=1}^{3} \int_{t_0}^{t} G_{ijk\ell}(r,t,\tau)\, d\sigma_{k\ell}(r,\tau), \quad \varepsilon_{ij}(r,t_0) = 0, \tag{2}$$

$$\varepsilon_{ij}(r,t) = \frac{1}{2}\left[\frac{\partial u_i(r,t)}{\partial x_j} + \frac{\partial u_j(r,t)}{\partial x_i}\right], \quad i,j = 1,2,3, \tag{3}$$

where t is a current time, $t_0 \leqslant t \leqslant T < +\infty$, $r = (x_1, x_2, x_3)$ is the vector of material (Lagrangian) coordinates of a material point of the system under investigation, $u(r,t) = \{u_i(r,t)\}$ is the vector of spatial displacements of material points as a function of the reference position, r, and t, $\epsilon(t) = \{\varepsilon_{ij}(r,t)\}$ is the strain tensor, and $\sigma(r,t) = \{\sigma_{ij}(r,t)\}$ is the stress tensor. Correspondingly, $F_{ijk\ell}(r, t, \tau)$ is the tensorial creep function and $G_{ijk\ell}(r, t, \tau)$ is the tensorial relaxation function. These kernels of integral operators depend on t and τ separately but not on difference $t - \tau$ and describe a relationship of the current stress with the whole time history of strain. After transformation of PIDE or SPIDE into OIDE or SOIDE, a time-dependent structure of the kernels is preserved in target equations of motion.

An important part of studies in stochastic viscoelasticity includes: (i) an analysis of a weak solvability of initial-boundary value problems [10], (ii) an examination of a stochastic stability [9,11], (iii) solution of reliability problems, etc.

It is well-known, that the solution of integral differential equations (IDE) is a very difficult problem even for the deterministic case. These difficulties are even greater for the linear and nonlinear stochastic cases. In spite of the existence of a few results concerning solvers for SOIDE (see for instance, [7]), it is very useful to adapt the existing methods for solving deterministic IDE to the stochastic case, because the main part of methods for qualitative and quantitative analysis of phenomena described by SOIDE consists of deterministic schemes.

A number of schemes have been developed for constructing numerical approximations for solution of deterministic and stochastic IDE. As for examination of stochastic problems, approximate algorithms are usually used for direct generation of time histories. Among these schemes there are:

(i) purely numerical methods (classical and modified hybrid variants) such as explicit and implicit (backward) Euler schemes, a discontinuous Galerkin method, energy methods [12], general, standard and Galerkin finite element methods for PIDE, the Tau method [13], the one-step Runge–Kutta and multi-step methods [14–16], the Runge–Kutta method [17] for calculation of covariance functions, extrapolation methods [18], Galerkin methods [19], the method of iterations at the last step, a usage of wavelets [20], globally defined Sinc basis functions [21], an approximate transformation of SOIDE into SODE on the base of replacements of kernels with respect to second arguments by piecewise constant functions [7,22], and gamma-distribution expansions;

(ii) approximate analytic methods, including methods of Taylor series [23], the successive approximation method for the computation of the Green function [24], the asymptotic method [25], the stochastic averaging method, the collocation method [26], the perturbation theory [27].

The main objective of this paper is to present a new tool for a statistical estimation of the solution of a class of IDE with random parameters and Gaussian stochastic processes as input. Stochastic linear viscoelastic problems are described by this class. The paper explains a theoretical apparatus and an algorithmic part of an application of the Green matrix-valued functions for time-domain analysis of systems of linear SOIDE effected by Gaussian nonstationary noises in the presence of model parameter uncertainties that are described in the framework of the probability theory.

This paper is organized as follows. Section 2 deals with the formulation of the problem under investigation. Section 3 is devoted to the construction of the equations for calculating the first- and second-order conditional moment functions of the stochastic solution. In Section 4, we give a short review of an application of the Green matrix-valued function for calculation of solution of linear IDE, and schemes for its approximate computation. Section 5 yields relations for Green matrix-valued functions for the first- and second-order conditional moment functions, and formal schemes for calculation of the conditional vector-valued mean function and the conditional matrix-valued covariance function. Section 6 deals with some approximations for the first- and second-order conditional moment functions obtained on the base of Green matrix-valued