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# Series representations for the rectification of a superhelix

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ABSTRACT

A superhelix is a curve that is helically coiled around a helix. Despite its importance in relation to the deformation modeling of various shapes, the superhelix is greatly overlooked, in part owing to its complexity and in part due to the lack of an analytical formula for its arc length. Deriving an exact analytical formula is not simple, because one needs to integrate a function without a closed-form integral solution to determine the arc length of a superhelix. In this study, we present a method by which to obtain the integral of the function that has no closed form integral by employing the series expansion approach of Maclaurin, as originally used to express the exact perimeter of an ellipse as an infinite sum. Our final expression of the arc length of a superhelix takes the form of two separate infinite sums, from which the one that converges is chosen to be applied, depending on the range of the geometric variables of the curve.



## 1. Introduction

A superhelix is a three-dimensional space curve that is helically coiled around a helix, as illustrated in Fig. 1. This particular curve is closely associated with what is known as supercoiling (also known as writhing), which is the helical coiling of an elastic rod subjected to a twisting load [1,2]. Superhelices can also represent geodesics on the surface of a cylinder subjected to three-dimensional bending, and hence can be used as deformation geometry for shape sensing of various deformable cylinder-shaped structures such as pipelines [3,4], endoscopes [5], biopsy needles [6], minimally invasive surgical instruments [7], and multicore optical fibers [8–10].

Measuring the shape of a deforming rod from surface strains has been given a considerable attention since Wilk [11]. In his work, Wilk shows detailed formulations by which to determine the deformation state of a cylindrical rod under bending, twisting, shear, and elongation. However, he does not cover the case where bending and twist take place at the same time, for doing that requires a proper geometric model for the strain on the surface of the rod. In recent years, Zhang et al. [4] and Froggatt et al. [9] introduced some approximate methods for the shape estimation of a bent and twisted rod. Froggatt's method proved particularly applicable for the helically wound multi-core optical fibers in which the strains are measured with very high resolution using the Rayleigh scattering. However, the deformation geometry proposed in these studies are still not the most exact models for a bent and twisted rod, and they also do not account for the Frenet–Serret torsion in three-dimensional bending. (Throughout this paper, the term torsion refers to the Frenet–Serret torsion. The twist of the material is termed as twist.)

When a cylindrical rod is bent in three-dimensional space with a constant curvature and a constant torsion, its central curve, which is the curve of the centers of the cross-sections, is bent into a helix. If the rod is also twisted at a constant

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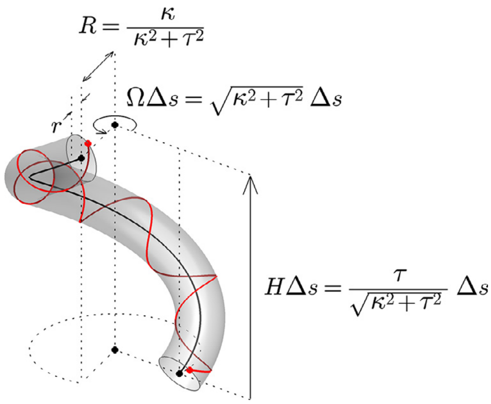


**Fig. 1.** A typical superhelix (red curve) bound around a helix (black curve). The length of the core helix is $\Delta s$. The grey surface depicts a cylindrical rod under three-dimensional bending with uniform curvature and torsion and which is hence deformed into a helical coil [16]. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article).

rate, the strain on the surface can be exactly modeled by using the arc length of a superhelix. Especially with strain sensors laid in a helical configuration, as in Froggatt et al. [9], a superhelix is an ideal geometric model for precise deformation and sensor strains. Moreover, even when the sensors are aligned only in straight longitudinal lines and no twist deformation is assumed, as in Childers et al. [10], a superhelix can still be useful as a deformation model. This is particularly true when a rod deforms in three-dimensional space and constantly changes its bending direction due to Frenet–Serret torsion. Eventually this causes the longitudinal lines to bend into superhelices, as noted by Yamada and Hirose [12]. This effect of Frenet–Serret torsion is also implied by Ericksen [13] and Shield and Im [14]. Here, we refer to this particular case of a superhelix with the rate of twist $\omega$ equal to 0 as an untwisted superhelix.

One of the difficulties when modeling with a superhelix is that you can not express its arc length in a simple formula, beacause doing so requires integration of a function that has no closed form integral, as presented in (3). It is of course possible to evade this problem by simply using numerical integration, but this approach is generally of inferior quality compared to using an exact analytical formula in terms of both accuracy and efficiency. An analytical formula can be of great advantage in cases where both accuracy and speed are highly demanded, one example of which is the real-time shape sensing of a rod under large dynamic deformation.

However, a similar issue with integration arose before in classical mathematics when attempting to formulate the perimeter of an ellipse. To the best of our knowledge, it was Colin Maclaurin who first found the exact expression for the perimeter of an ellipse as an infinite sum, by expanding the integral kernel into an infinite series first, and then integrating the general term of the series [15]. Inspired by this method, we derived a formula for the arc length of a superhelix as an infinite sum in our previous work [16], but it was an incomplete formula that we can only employ within a limited range of the geometric variables that satisfies $|1 - r\kappa \cos((\omega - \tau)s + \psi)| \le |r\omega|$. Since then we have been able to derive another series representation that we can use in cases where $|1 - r\kappa \cos((\omega - \tau)s + \psi)| > |r\omega|$. In this paper, we present the derivation of the two series, (14) and (25), using either of which we can determine the exact arc length of any superhelix.

## 2. The arc length of a superhelix

In our previous work [16], we derived the parametric equation of a superhelix that coils around a helix of curvature $\kappa$ and torsion $\tau$ with radius $r$, rotational rate $\omega$, and phase angle $\psi$ as follows:

$$\mathbf{F}(s) = [F_1(s)\ F_2(s)\ F_3(s)]^{\mathrm{T}}$$
$$= \begin{bmatrix} \frac{\kappa}{\kappa^2+\tau^2}\cos\left(\sqrt{\kappa^2+\tau^2}s\right) - r\cos((\omega-\tau)s+\psi)\cos\left(\sqrt{\kappa^2+\tau^2}s\right) \\ +r\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\sin((\omega-\tau)s+\psi)\sin\left(\sqrt{\kappa^2+\tau^2}s\right) \\ \\ \frac{\kappa}{\kappa^2+\tau^2}\sin\left(\sqrt{\kappa^2+\tau^2}s\right) - r\cos((\omega-\tau)s+\psi)\sin\left(\sqrt{\kappa^2+\tau^2}s\right) \\ -r\frac{\tau}{\sqrt{\kappa^2+\tau^2}}\sin((\omega-\tau)s+\psi)\cos\left(\sqrt{\kappa^2+\tau^2}s\right) \\ \\ \frac{\tau}{\sqrt{\kappa^2+\tau^2}}s + r\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\sin((\omega-\tau)s+\psi) \end{bmatrix}. \tag{1}$$

The arc length of $\mathbf{F}(s)$ within $s_1 \le s \le s_2$ is given by

$$L|_{s_1}^{s_2} = \int_{s_1}^{s_2}\sqrt{\left[\frac{\mathrm{d}}{\mathrm{d}s}F_1(s)\right]^2 + \left[\frac{\mathrm{d}}{\mathrm{d}s}F_2(s)\right]^2 + \left[\frac{\mathrm{d}}{\mathrm{d}s}F_3(s)\right]^2}\,\mathrm{d}s$$