ARTICLE IN PRESS
JID: APM [m3Gsc;January 30, 2018;20:49]

Applied Mathematical Modelling 000 (2018) 1–16

Contents lists available at ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

On the hyperbolicity of the two-fluid model for gas–liquid bubbly flows

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ARTICLE INFO

Article history:
Received 6 June 2017
Revised 28 December 2017
Accepted 8 January 2018
Available online xxx

Keywords:
Two-fluid model
Hyperbolicity
Bubbly flows
Bubble dispersion
Shock tube
Bubble column

ABSTRACT

The hyperbolicity condition of the system of partial differential equations (PDEs) of the incompressible two-fluid model, applied to gas–liquid flows, is investigated. It is shown that the addition of a dispersion term, which depends on the drag coefficient and the gradient of the gas volume fraction, ensures the hyperbolicity of the PDEs, and prevents the non-physical onset of instabilities in the predicted multiphase flows upon grid refinement. A constraint to be satisfied by the coefficient of the dispersion term to ensure hyperbolicity is obtained. The effect of the dispersion term on the numerical solution and on its grid convergence is then illustrated with numerical experiments in a one-dimensional shock tube, in a column with a falling fluid, and in a two-dimensional bubble column.

1. Introduction

The two-fluid model [1–5] probably represents the most widely adopted approach to describe the spatial and temporal evolution of gas–liquid flows in systems of practical relevance, due to its moderate computational cost. For this reason, the correct formulation of the model has been subject of several studies, which aimed, on one hand, to ensure the desired mathematical property of hyperbolicity of the model equations, and, on the other hand, to appropriately incorporate the description of physical phenomena experimentally observed in bubbly flows. In particular, the numerical stability of the solution obtained from the two-fluid model depends on the characteristics of the underlying equations, which, as shown in [6–8], may be complex. In such a case, the discretized equations do not allow a grid-converged solution to be achieved, and unstable modes in the solution appear, severely affecting the model prediction and its sensitivity to grid refinement. Several approaches were suggested in the literature to address this problem. Stuhmiller [6] observed that the addition of a certain amount of dissipation mitigates the problem for a specific grid resolution. The problem of complex characteristics was addressed in [9] with the introduction of surface tension effects. A criterion for the grid resolution that ensures the well-posedness of two-fluid problem, relating the minimum grid size to a multiple of the bubble radius, was proposed in [8].

It was demonstrated in [10] that real characteristics of the two-fluid equations are a necessary condition in order not to violate the causality requirement. In particular, his work shows that, when the two-fluid equations have complex

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https://doi.org/10.1016/j.apm.2018.01.011

characteristics, it is necessary to know all the values at the solution at future times $t > t_i$, in order to determine an accurate solution at an arbitrary time t_i. Such a result clearly violates the physical constraint of causality, and highlights the importance of ensuring that the mathematical model is hyperbolic. Based on these observations, several researchers proposed modified version of the two-fluid model that guarantee hyperbolicity under certain conditions. Some authors [11–13] performed numerical regularization of the ill-posed equation, which, however, comes at the expense of accuracy, since it relies on the addition of numerical dissipation.

Several investigators [6,9,14–19] ensured the hyperbolicity of the two-fluid model by introducing a pressure term in the phase momentum equation. The presence of the same mean bulk pressure P_k in both the phases was assumed in [6], and added an interface pressure term $(p_{\mathrm{ki}} - P_{\mathrm{k}})\nabla\alpha$, where p_{ki} is the interface pressure for phase k, in order to make the two-fluid model hyperbolic. He considered the interface pressures p_{ki} for both the phases to be equal, and modeled them based on the analytical solution of pressure distribution around an isolated sphere [20]. An interface pressure term $p_{\mathrm{i}}\nabla\alpha$ was used in [14], where, differently from [6], p_i is a coefficient determined to ensure the hyperbolicity of the set of equations, without physical justification. The addition of this term, however, was deemed controversial [12], as shown by Sha and Soo [21].

A study of wave propagation in adiabatic mono-disperse bubbly flows was performed in [22], who concluded that the interfacial pressure difference and the gradients of the void fraction in the non-drag momentum exchange terms dominate the behavior of void wave propagation in bubbly flows.

Real characteristics for the two-fluid model equations were ensured by assuming the mean bulk pressures in both the phases to be different in [9]. These pressures were related through the Laplace constraint, and their difference is set to be proportional to the surface tension. In other works [15–19], the two-fluid model is made hyperbolic using a two-pressure formulation for compressible two-phase flows, where pressures in each phases are computed using an equation of state. However, these authors follow different approaches to model the interface pressure term $(p_{\mathrm{ki}} - P_{\mathrm{k}})\nabla\alpha$. The interface pressures p_{ki} were assumed to be equal in [15,16]. Their model is valid only for stratified flows, and it requires an additional transport equation for the volume fraction in terms of the interface velocity to close the set of equations. Similarly, equal interface pressures were considered in [17], and calculated as a function of the mixture pressure. The interface pressure coefficient $(p_{\mathrm{ki}} - P_{\mathrm{k}})$ was modeled in both the phases in terms of the surface tension and bulk modulus by Saureland Abgrall [17], Chung [23] and Jung et al. [24].

Wave propagation analysis in bubbly flows was performed in [25,26], where it was found the virtual mass term has a significant effect on the dispersion of waves in these flows. The existence of complex characteristics in the two-fluid model with virtual mass term was confirmed by Chung et al. [19] and Thorley and Wiggert [27]. These authors proposed the introduction of an interfacial pressure jump term, directly proportional to the surface tension between the phases, in order to ensure the hyperbolic behavior of the two-fluid model. The resulting model with virtual mass and interfacial pressure jump was hyperbolic, but the wave dispersion may become excessive, depending on the value of the coefficient used in the virtual mass model. A detailed study of the pressure forces in disperse two-phase flows can be found in [28]. The stability of a two-fluid model containing only first-order differential terms and algebraic closures was studied in [29], showing that the stability properties of the model do not depend on the wavelength of the perturbation, which is unphysical. The study was extended in [28] to incorporate the effect of terms with derivatives of arbitrary order. However, the authors of this study concluded that the introduction of these terms is ineffective at improving the long-wavelength stability of the hyperbolic two-fluid model containing only first-order differential terms.

The stability of a uniform suspension of bubbles was examined in [30], identifying a critical value of volume fraction below which the suspension is stable. This result is confirmed experimentally in [31], where experiments of air injection in bubble columns were performed. The standard two-fluid model, however, predicts flow instabilities also in these conditions, showing an unphysical behavior.

The interfacial momentum transfer term in the two-fluid model was described including a dispersion term proportional to the drag coefficient and to the gradient of the gas volume fraction in [32], where an interfacial pressure jump based on the work of Stuhmiller [33] and Pauchon and Banerjee [6] was used. The dispersion coefficient was defined in [32] as a function of the turbulent eddy viscosity. The hyperbolicity of the two-fluid equations was ensured in [12] by modifying the virtual mass coefficient as a function of the volume fraction and of the density ratio of the phases, obtained assuming the multiphase mixture is incompressible. Hyperbolicity and stability of the two-fluid model were studied in [34] in terms of the momentum flux parameters they introduced in the model to incorporate the effect of void fraction and phase velocities. The hyperbolicity condition was then determined to identify when the model equation are *stable*, in a mathematical sense, for specific flow conditions.

The hyperbolicity of a two-fluid model for gas–particle flows was investigated in [35], where a modification of the form of the buoyant term to incorporate the effect of the relative motions of the phases, leading to hyperbolic equations. This development was based on the observation made in [36], where the origin of the complex characteristics of the two-fluid equations was attributed to the buoyancy term. However, for gas–particle systems the compressible particle phase has a separate pressure, function of the granular temperature and of the frictional pressure, which is often sufficient to ensure the conditional hyperbolicity of the two-fluid model for an interval of volume fractions [35,37].

Lhuillier et al. [38] investigated the well-posedness of the six-equation two-fluid model for dispersed mixtures, with the ultimate goal of obtaining a hyperbolic form of the model. They showed that the two-fluid model with equal pressures satisfies the energy conservation constraint for the mixture. They found that the model for the interfacial pressure proposed by Stuhmiller [6] does not respect this constraint, and, consequently is not physical. However, they observed the importance