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A reduced model to locate low contrast inclusions in poroelastic media

J. Mura

Biomedical Imaging Center, Pontificia Universidad Católica de Chile, Santiago, Chile

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ABSTRACT

This paper introduces a numerical method to localize inclusions having slightly different elastic coefficients than those of a fully saturated poroelastic matrix, whose detection is often difficult. This method can be used to find weakly stiffer or softer objects in saturated soils or diseased biological tissues at early stages. To this end, we propose a reduced model from the Biot's equations by splitting the fluid pressure into two parts: one embedded into an elasticity model and the other one used as a corrector term. By applying the small amplitude homogenization method, we can successfully retrieve the position and extension of inclusions in poroelastic media employing this simplified model. Numerical results show a good agreement for the location of inclusions when the contrast is below 30% stiffer or softer than the matrix, and for a noise level up to 5% for frequencies below 50 Hz.

1. Introduction

The non-invasive detection of inclusions is a very active field because it permits to find objects of interest without destroying the specimen. There is an extended range of applications in porous media, being of the utmost importance in engineering to discover subsoil properties in geophysical explorations, when the extension of the domain is too vast to consider multiple direct digging procedures. Other applications are the detection of failure zones in wet structures, early detection of pathologies in biological tissues, the identification of mineral or hydrological resources, among many others.

In particular, some advances in locating inclusions in porous media can be found in [1], where shape derivatives and sensitivity analysis are used to discover these inclusions in a quasi-static linear poroelastic regime. A reduced model in the incompressible limit is introduced in [2], where Young modulus can be estimated, by using an overlapped sub zone technique proposed in [3]. More recently, some parameters are obtained for a fixed number of horizontal layers using the Biot theory in time harmonic regime, as shown in [4]. Other approaches include, as in [5], the explicit deduction of drained constants from the undrained ones in the context of acoustic waves in homogeneous poroelastic media.

Since it is not possible, in general, to ensure uniqueness of solutions (e.g., [6]), the detection and localization of special features for most of these techniques is carried out by minimizing the misfit between experimental and simulated data. Additionally, when the contrast between properties of materials that compose the inclusions and their surrounding media is low, the detection is less feasible.

In contrast to the approaches mentioned above, the small amplitude homogenization (SAH) takes advantage on the slight variation in coefficients of interest orin entire elastic tensors by performing an asymptotic expansion for a given contrast parameter. Introduced by L. Tartar as a particular case in the Homogenization theory [7], the SAH has been applied to optimal

E-mail addresses: jamura@uc.cl, joaquin.mura@gmail.com

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design (see [8–10]) and inverse problems in elastostatic and elastodynamics [11–13]. The idea is to construct a minimizing sequence of characteristic or indicator functions, providing an implicit localization of the inclusions using discontinuous coefficients into the mathematical model. That minimization is performed via an asymptotic expansion up to second order with respect to the contrast parameter, without any restriction on the shape, size, location or amount of inclusions.

In this work, we introduce a novel approach to solve the problem of detection and localization of elastic inclusions in poroelastic media. Based on a pressure decoupling strategy, the steady-state Biot's model is written as the combination of an elasticity problem (with undrained coefficients), and a pressure model (for the fluid contributions). Under the assumption of low contrast in elastic coefficients, the SAH method is used to conceive a sequence of reduced complex-valued problems. This method allows the construction of an efficient optimization method to identify low-contrast inclusions from data acquired in poroelastic media, avoiding the resolution of full poroelastic problems at each iteration without sacrificing feasibility.

The paper is organized as follows. The full Biot's model in harmonic time regime is introduced in Section 2. The discussion about the reduced method is detailed in Section 3. The relaxation method using small amplitude homogenization and their corresponding gradient calculations are explained in Section 4. Numerical examples are shown and discussed in Section 5, showing the performance and reliability of this method. Concluding remarks are stated in Section 6.

2. The Biot model

2.1. Fully saturated model for a single pore fluid phase

The Biot's model takes into account the relation between the solid displacement $\boldsymbol{u}$, the pore pressure p and the averaged Darcy velocity $\boldsymbol{v}$ of the percolating fluid, also known as flow velocity. The fluid displacement relative to the solid matrix displacement is $\boldsymbol{w}$, and it is related to the velocity through $\boldsymbol{v} = \dot{\boldsymbol{w}}$. In the $\boldsymbol{u} - \boldsymbol{w} - p$ formulation, the relation among these variables satisfies

$$\begin{aligned} &\rho\ddot{\boldsymbol{u}} + \rho_f\ddot{\boldsymbol{w}} - \mathrm{div}\sigma = \rho\boldsymbol{g} \\ &\kappa\nabla p + \dot{\boldsymbol{w}} + \rho_f\kappa\ddot{\boldsymbol{u}} + \frac{\rho_f}{\phi}\kappa\ddot{\boldsymbol{w}} = \rho_f\kappa\boldsymbol{g} \\ &\mathrm{div}\dot{\boldsymbol{w}} + \alpha\mathrm{div}\dot{\boldsymbol{u}} + \frac{\dot{p}}{Q} = 0. \end{aligned} \tag{1}$$

with the linear elastic constitutive, also known as Hooke law

$$\sigma = \mathrm{C}e - \alpha p\mathbf{I} \tag{2}$$

relating the total stress tensor σ with the linear strain $e(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T)$, and the pore pressure. The solid properties are stored in C, the elasticity tensor. In the particular case of isotropic materials, the tensor is described in terms of the Lamé coefficients as

$$\mathrm{C}_{ijkl} = \mathrm{C}_{ijkl}(\lambda, G) = \lambda\delta_{ij}\delta_{kl} + G\big(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\big), \tag{3}$$

where δ_{ij} is the Kronecker delta (see [14]).

The parameter $\alpha > 0$, known as the Biot-Willis coefficient, physically corresponds to the ratio of the incremental volume of fluid divided by the change in bulk volume under constant pore pressure (see [14]). In the right-hand side of (1), g is the gravity acceleration constant. The permeability of the fluid is denoted by κ, and the effective density is averaged by the fluid (ρ_f) and solid (ρ_s) parts, this is $\rho = \phi\rho_f + (1-\phi)\rho_s$. Finally, the Biot parameter Q is given by

$$\frac{1}{Q} = \frac{\phi}{K_f} + \frac{\alpha - \phi}{K_s} \tag{4}$$

where K_s and K_f are the solid and fluid bulk modulus, respectively. (See references [14,15] or [16] for further details.)

2.2. The time harmonic case

Assuming a monochromatic pulsation regime, for any real-valued function $f = f(\boldsymbol{x}, t)$, the complex amplitude $\hat{f}$ of f is given by

$$f(\boldsymbol{x}, t) = \Re\mathbf{e}(\hat{f}(\boldsymbol{x}, \omega)e^{i\omega t}).$$

For the sake of simplicity, we will denote harmonic regime variables without the hat-symbol, except their coefficients. Applying the definition given above into the system (1) yields