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Multistability and fast-slow analysis for van der Pol–Duffing oscillator with varying exponential delay feedback factor

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ABSTRACT

Time delays are many sources of complex behavior in dynamical systems. Yet its relationship with bursting dynamics needs to be further explored, particularly when the strength of feedback is a nonlinear function of delay. In this paper, we analyze the dynamics of the van der Pol–Duffing fast-slow oscillator controlled by the parametric delay feedback, where the strength of feedback control is a function exponential varying with the time delay. The system may exhibit a unique equilibrium point and three ones for the different parameters by employing the pitchfork bifurcation. Next, the stability-switches and the Hopf bifurcation curves are presented as the delay varies, which leads to the occurrence of novel bursting phenomena. Some weak resonant or non-resonant double Hopf bursting oscillations are presented due to the vanishing of real parts of two pairs of characteristic roots. Not only the magnitude of the time delay itself but also the strength of feedback control may influence the dynamical evolution process of bursting behaviors in the delayed system. Such fast-slow forms about bursting dynamics, as well as classifications about local dynamics are investigated. Furthermore, periodic and quasi-periodic bursting motions are verified in both theoretical and numerical ways.

1. Introduction

As a typical kind of second-order nonlinear dynamic system, van der Pol–Duffing equation is one of the commonest examples in research articles, which describes the oscillations in vacuum tube circuit [1–3]. Many efforts have been made to find its analytical solution or construct Poincaré maps to illustrate its important dynamical features [4–6]. In reality, the time delay is inevitable such as physical systems, manufacturing process, population dynamics, controlling systems and network communication systems, and the delay feedback control has been widely applied in mechanical and electronic facilities [7–10].

Our study aims at the dynamical behaviors of the van der Pol–Duffing oscillator with a parametric delay feedback, which can be described as

$$\ddot{x}(t) + \delta\left[x^2(t) - a\right]\dot{x}(t) + \alpha_1 x(t) + \alpha_2 x^3(t) = A\left[e^{-p\tau}x(t-\tau) - x(t)\right] \tag{1}$$

where δ, a, and α_1 are positive real constants, and $\tau > 0$ is the time delay. Note that the strength of feedback control takes the form of $Ae^{-p\tau}$, i.e., a function exponential varying with the time delay. This implies that the feedback effect of the

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past state is changing with the time. Apparently, the strength of delayed feedback control becomes the delay-independent constant for $p = 0$. Here p is called as the exponent change rate of feedback control.

Eq. (1) becomes the van der Pol–Duffing system with a regular delay at $p = 0$, which has served as one of many basic models in physics, electronics, biology, neurology and so on [11–16]. Early results about van der Pol–Duffing equation with delay are concerned with approximate solutions and the normal form computation [17,18]. For example, the center manifold method has been adopted to reduce this parametrical time delay oscillator into finite-dimensional system, and the bifurcation analysis could be achieved for the reduced system including Hopf bifurcations, double Hopf bifurcations and other complex dynamics [19–21].

For our study, the exponent change rate is excited by a slow-varying item, as it takes the form of $p = B\cos(\Omega t)$, where B is the excitation amplitude, Ω is the excitation frequency and set as $0 < \Omega \ll 1$. Noting that the order gap exists between the excitation frequency and the natural frequency of system without excitation, the effect of multiple time scales appears [22–24]. This leads to the emergence of bursting patterns [25,26], where the slow varying rate p plays an important role in dynamical evolution process.

Bursting oscillations are periodic orbits of a dynamical system characterized by an alternation between oscillations of very distinct large and small amplitudes. These types of dynamic were reported in neuronal recordings, mechanical and electrical systems [27–29]. Moreover, bursting oscillations have also been linked to many dynamical mechanisms, such as the singularity of Hopf bifurcation, break-up of invariant torus and slow passage effect through various bifurcation behaviors [30].

However, to the best of our existing knowledge, there are still few articles about higher codimensional bursting phenomena induced by the parametric delay feedback. Our present work aims at investigating the occurrence of possible complex bursting oscillations in van der Pol–Duffing oscillator with exponential delay-dependent parameters. We shall use the bifurcation analysis method in delay differential equations (DDEs) introduced by Stepan and co-workers [31–34] to investigate the stability of the fixed point and the dynamics near the equilibrium point under the Hopf bifurcations. This dynamical mechanism may yield complex oscillatory patterns including local cycles, relaxation cycles or transitions with torus solutions, where we can characterize with a combination of bifurcation analysis and numerical simulations.

The paper is organized as follows. After introducing our model and the motivations behind this work in Section 1, we investigate the two-parameter geometrical criterion to the stability and the Hopf bifurcations of the fast system corresponding to Eq. (1) in Section 2. The following bursting dynamics can be analyzed as a function of the exponent change rate. In Section 3, bursting oscillations switching around multiple stable states are presented in this model, where the change rate can be considered as a tuning parameter to regulate such bursting dynamics. Section 4 shows the existence of resonant or non-resonant double Hopf bifurcation and how these elements are the source of complex oscillatory patterns with quasi-periodic spiking motions. Finally, Section 5 summarizes the main conclusions of the paper.

2. Stability switches and bifurcations on slow manifold

In this section, we first consider the system (1) as the coupling of two autonomous subsystems by regarding the change rate of p as a slow subsystem (or variable), which is written as $p = B\cos(\Omega t)$. The complex bursting dynamics can be analyzed through the fast subsystem **(FS)** of Eq. (1). The fast subsystem may dominate the dynamics while the slow subsystem (variable) may modulate the behaviors of FS under multiple time scale effect. Therefore, we begin our study at investigating the stability-switches and the Hopf bifurcations of the fast subsystem, where the delay τ and the exponent rate p have been chosen as bifurcation parameter.

2.1. Analysis for equilibrium points

Simple in form as it is, the equilibrium points of FS can be written in the form of $E(x, y) \equiv E(x, \dot{x}) = E(x_0, 0)$, where x_0 is decided by the real roots of the following equation:

$$-\alpha_1 x_0 - \alpha_2 x_0^3 + A\left(e^{-p\tau} x_0 - x_0\right) = 0, \text{ i.e., } x_0\left(-\alpha_1 - \alpha_2 x_0^2 + Ae^{-p\tau} - A\right) = 0 \tag{2}$$

From Eq. (2), it is easy to see that origin is the unique equilibrium when $(-\alpha_1 + Ae^{-p\tau} - A)/\alpha_2 < 0$; whereas for $(-\alpha_1 + Ae^{-p\tau} - A)/\alpha_2 > 0$, system (1) has three equilibrium points $E_0(0,0)$ and $E_{\pm}(\pm\sqrt{(-\alpha_1 + Ae^{-p\tau} - A)/\alpha_2}, 0)$, the local stability of which can be determined by using the associated characteristic equations. Moreover, by using local bifurcation analysis method, we can conclude that the three equilibrium points may join together to form a cusp bifurcation behavior, where a pitchfork bifurcation occurs at $-\alpha_1 + Ae^{-p\tau} - A = 0$, corresponding to the colliding between the two symmetric nontrivial equilibrium points and the trivial equilibrium point.

More precisely, in case of $\alpha_2 > 0$, if $-\alpha_1 + Ae^{-p\tau} - A < 0$, the system has unstable trivial equilibrium points E_0 and the other two stable equilibrium points; while if $-\alpha_1 + Ae^{-p\tau} - A > 0$, the untrivial equilibrium points disappear while the trivial equilibrium point still exists but it becomes stable. On the other hand, in case of $\alpha_2 < 0$, if $-\alpha_1 + Ae^{-p\tau} - A < 0$, the system has only unstable trivial equilibrium point E_0; while if at the condition of $-\alpha_1 + Ae^{-p\tau} - A > 0$, the trivial equilibrium point still exists but it becomes stable, and meanwhile the other two unstable $E_{\pm}$ appear and coalesce at zero (see in Fig. 1).