Probabilistic Engineering Mechanics 45 (2016) 61–69

Contents lists available at ScienceDirect

Probabilistic Engineering Mechanics

journal homepage: www.elsevier.com/locate/probengmech

Analysis of complex system reliability with correlated random vectors

Wenliang Fan [a,b,c,*], Pengchao Yang [b], Alfredo H-S. Ang [c], Zhengliang Li [a,b]

[a] Key Laboratory of New Technique for Construction of Cities in Mountain Area (Chongqing University), Ministry of Education, Chongqing 400045, China
[b] School of Civil Engineering, Chongqing University, Chongqing 400045, China
[c] Department of Civil and Environmental Engineering, University of California, Irvine, CA 92697, USA

ARTICLE INFO

Article history:
Received 22 March 2015
Received in revised form
4 March 2016
Accepted 30 March 2016
Available online 1 April 2016

Keywords:
System reliability
Correlated random vectors
Transformation
Taylor series expansion
Complete system failure process
Fourth-moment method

ABSTRACT

The analysis of reliability of complex engineering systems remains a challenge in the field of reliability. It will be even more difficult if correlated random vectors are involved, which is generally the case as practical engineering systems invariably contain parameters that are mutually correlated. A new method for transforming correlated distributions, involving the Nataf transformation, is proposed that avoids the solution of integral equations; the method is based on the Taylor series expansion of the probability density function (PDF) of a bivariate normal distribution resulting in an explicit polynomial equation of the equivalent correlation coefficient. The required numerical results can be obtained efficiently and accurately.

The proposed method for transformation of correlated random vectors is useful for developing a method for system reliability including complex systems with correlated random vectors. Based on the complete system failure process (originally defined as the development process of nonlinearity) and the fourth-moment method, the analysis of system reliability for elastic-plastic material avoids the identification of the potential failure modes of the system and their mutual correlations which are required in the traditional methods. Finally, four examples are presented – two examples to illustrate the potential of the new method for transformation of correlated random vectors, and two examples to illustrate the application of the proposed more effective method for system reliability.

1. Introduction

Traditionally, assessing the reliability of structural systems requires the identification of the major failure modes and the associated probability analysis of the correlated failure modes. If correlated random variables among the structural parameters and loads are involved, both the analysis of failure probabilities for the respective failure modes in identifying the dominant failure modes and evaluation of the correlation coefficients among the different failure modes in probability analysis become more complicated. For this reason, studies on system reliability generally avoid considering systems with correlated parameters. In order to estimate the system reliability with correlated variables and evaluate the impact of correlations among the components of a random vector on the reliability of a system, both a practical approach for transformation of correlated vectors and an effective method for assessing the reliability of complex systems are necessary.

Mutual correlations among random vectors generally exist in stochastic systems. Proposed are alternative and more effective methods for the transformation of correlated random vectors, and its application in a more effective reliability assessment of complex systems including systems with correlated random parameters.

The present paper is organized as follows. In Section 2 a new method is proposed for tackling the correlation-remodeling problem in the normal-to-nonnormal transformation. Then, by combining the *complete system failure process* (based on the development process of nonlinearity by Chen and Li [1]) with the fourth-moment method [2] and the proposed transformation method of Section 2, a practical approach for system reliability with correlated variables is presented in Section 3. To verify the accuracy and rationale of the proposed formulations, four examples are numerically examined in Section 4. Finally, Section 5 summarizes some conclusions.

2. On transformation of correlated random vectors

The reversible transformation, between a correlated random vector and its corresponding independent normal random vector, could provide an effective procedure for different cases of known probabilistic information of the correlated random vectors. In the

* Corresponding author at: School of Civil Engineering, Chongqing University, Chongqing 400045, China.
E-mail address: davidfwl@126.com (W. Fan).

http://dx.doi.org/10.1016/j.probengmech.2016.03.004
0266-8920/

simplest case in which the components of a random vector are statistically independent, the one-dimensional Rosenblatt transformation [3] is adequate. For the second case with known joint PDF, the multi-dimensional Rosenblatt transformation is suitable; however, the joint PDF is often difficult to obtain or is seldom available. For this latter case, if the marginal distributions and fixed correlation matrix are known, which is a common situation in practical applications, an effective and efficient transformation is the normal-to-nonnormal vector transformation [4–9], known as the Nataf transformation in the field of reliability and random analysis [10,11]. In general, this involves two processes; the first one transforms the targeted correlated random vector into a correlated normal random vector; and the second process transforms the correlated normal random vector into an independent normal random vector. The second process can be performed with the Cholesky decomposition. Therefore, the crucial part of this method is to specify the equivalent correlation matrix of the correlated normal random vector, which is equivalent to solving complicated integral equations.

Define $\boldsymbol{\Theta}=(\Theta_1,\Theta_2,\ldots,\Theta_n)^T$ as a random vector with the marginal cumulative distributions $F_1(\bullet)$, $F_2(\bullet)$,..., $F_n(\bullet)$ and the correlation matrix $\boldsymbol{\rho}$, and $\boldsymbol{X}=(X_1,X_2,\ldots,X_n)^T$ is the corresponding standard normal random vector with the correlation matrix $\boldsymbol{\rho}^*$. Obviously, Θ_i and Θ_j, X_i and X_j, and the elements of $\boldsymbol{\rho}$ and $\boldsymbol{\rho}^*$, say ρ_{ij} and ρ^*_{ij}, should satisfy the following equations

$$\Theta_i = F_i^{-1}\big(\Phi(X_i)\big) \tag{1a}$$

$$\begin{aligned}\rho_{ij} &= \frac{E\Big[\big(\Theta_i-\mu_i\big)\big(\Theta_j-\mu_j\big)\Big]}{\sigma_i\sigma_j}\\ &= \frac{E\Big[\Big(F_i^{-1}\big(\Phi(X_i)\big)-\mu_i\Big)\Big(F_j^{-1}\big(\Phi(X_j)\big)-\mu_j\Big)\Big]}{\sigma_i\sigma_j}\\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{F_i^{-1}\big(\Phi(x_i)\big)-\mu_i}{\sigma_i}\cdot\frac{F_j^{-1}\big(\Phi(x_j)\big)-\mu_j}{\sigma_j}\cdot\\ &\quad\varphi_2\big(x_i,x_j,\rho^*_{ij}\big)\mathrm{d}x_i\mathrm{d}x_j\end{aligned} \tag{1b}$$

$$\big|\rho_{ij}\big| \le \big|\rho^*_{ij}\big| \quad \text{and} \quad \rho_{ij}\cdot\rho^*_{ij}\ge 0 \tag{1c}$$

where $E[\bullet]$ is expectation operator, μ_i, μ_j and σ_i, σ_j are the means and standard deviations of Θ_i and Θ_j , respectively, $F_i^{-1}(\bullet)$ is the inverse function of $F_i(\bullet)$, $\Phi(\cdot)$ is the cumulative distribution function (CDF) of the standard normal variable, $\varphi_2(\cdot)$ is the bivariate standard normal PDF with correlation coefficient ρ^*_{ij}. In fact, Eq. (1b) can be viewed as a correlation-remodeling problem, namely remodeling the original correlation coefficients ρ_{ij} by the equivalent correlation coefficients ρ^*_{ij}.

Because of symmetry, $\boldsymbol{\rho}^*$ has $n(n\text{-}1)/2$ different elements which are determined by $n(n\text{-}1)/2$ different integral equations such as Eq. (1b). Since all the $n(n\text{-}1)/2$ equations are mutually independent, these equations can be solved either simultaneously or separately. In the simulation approach, all equations are solved by a stochastic root-finding approach simultaneously [8,9]. But the Cholesky decomposition may fail during the process of root searching, and the simulation approach may be more expensive than other methods. Comparatively speaking, the separate approach is more common. Generally, there are three kinds of methods to solve Eq. (1b) separately, namely by analytical methods [12,13], semi-empirical formulas [10,14] and by numerical methods [4,6,8,15,16]. Analytical method gives the exact formula of ρ^*_{ij} in terms of ρ_{ij} directly, but only works for some special random vectors, such as a uniformly distributed or lognormal vector. Analogously, a semi-empirical formula provides an approximate expression of ρ^*_{ij} for ρ_{ij}; however, it is also limited to specific distributions and coefficients of variation (cov) of the variables. With the combination of iterative methods for root searching and two-dimensional numerical integration methods, such as the Gaussian quadrature and Newton's integrations, the numerical method is most popular for correlation-remodeling problems, but is of low efficiency because numerous two-dimensional integrals are involved during the root-finding process. What's more, the numerical method may be inefficient and inaccurate if ρ^*_{ij} is close to 1 or -1. In references [15,16], another equation, equivalent to Eq. (1b), is formulated in the independent standard normal space, but two-dimensional integrals still exist and the result depends on the sequence of variables.

2.1. A new method to generate the correlation matrix of normal random vector

Introducing a numerical integration method into Eq. (1b), the integral equation on ρ^*_{ij} becomes an ordinary nonlinear equation as follows:

$$\begin{aligned}\rho_{ij} \approx \sum_{l=1}^{d_i}\sum_{m=1}^{d_j} w_{i,l}w_{j,m}&\left(\frac{F_i^{-1}\big(\Phi(x_{i,l})\big)-\mu_i}{\sigma_i}\right)\left(\frac{F_j^{-1}\big(\Phi(x_{j,m})\big)-\mu_j}{\sigma_j}\right)\\ &\varphi_2\big(x_{i,l},x_{j,m},\rho^*_{ij}\big)\end{aligned} \tag{2a}$$

or

$$\begin{aligned}\rho_{ij} = \sum_{l=1}^{d_i}\sum_{m=1}^{d_j}&\frac{w_{GH,l}w_{GH,m}}{\pi}\frac{F_i^{-1}\Big(\Phi\big(\sqrt{2}x_{GH,l}\big)\Big)-\mu_i}{\sigma_i}\\ &\frac{F_j^{-1}\Big(\Phi\big(\sqrt{2}x_{GH,m}\big)\Big)-\mu_j}{\sigma_j}\\ &\frac{\varphi_2\big(\sqrt{2}x_{GH,l},\sqrt{2}x_{GH,m},\rho^*_{ij}\big)}{\varphi\big(\sqrt{2}x_{GH,l}\big)\varphi\big(\sqrt{2}x_{GH,m}\big)}\end{aligned} \tag{2b}$$

where $x_{i,l}$, $x_{j,m}$ and $w_{i,l}$, $w_{j,m}$ ($l=1,\ldots,d_i$; $m=1,\ldots,d_j$) are the points and associated weights of a quadrature rule, $x_{GH,l}$ and $w_{GH,l}$ are the abscissas and weights of the Gauss–Hermite quadrature formula, d_i and d_j are the numbers of points on the X_i and X_j axes, respectively. Obviously, Eq. (2a) corresponds to a general quadrature rule and Eq. (2b) exploits the Gauss–Hermite quadrature. When the integrand in Eq. (1b) does not vary dramatically, the right side of Eq. (2) is a good approximation for the two-dimensional integration in Eq. (1b), and ρ^*_{ij} from Eq. (2) has high precision. However, when ρ^*_{ij} is close to 1 or -1, the bivariate normal PDF concentrates mostly on the domain of $x_i=x_j$ or $x_i=-x_j$, and the traditional numerical integration is inefficient and can be inaccurate.

2.1.1. Taylor series expansion of the bivariate standard normal PDF

The bivariate normal PDF of X_i and X_j with the correlation coefficient ρ^*_{ij} is expressed as

$$\begin{aligned}&\varphi_2\big(x_i,x_j,\rho^*_{ij}\big)\\ &\quad=\frac{1}{2\pi\sqrt{1-\rho^{*2}_{ij}}}\exp\left[-\frac{1}{2\big(1-\rho^{*2}_{ij}\big)}\big(x_i^2+x_j^2-2\rho^*_{ij}x_ix_j\big)\right]\end{aligned} \tag{3}$$

This PDF is also a function of ρ^*_{ij} and can be approximated by its Taylor series expansion on ρ^*_{ij}. The Taylor series expansion of Eq. (3) at the point $\rho^*_{ij}=0$ is