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Investigations on the bifurcation of a noisy Duffing–Van der Pol oscillator

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ABSTRACT

This study carried out investigations on the bifurcation characteristics of a Duffing–Van der Pol (DVDP) oscillator subjected to white noise excitations. Dynamical (or D-) bifurcations are characterised by dramatic changes in the dynamical behaviour leading to topological changes in the phase portrait. An additional mode of bifurcation - phenomenological (or P-) bifurcation, is observed in stochastically excited systems when the stationary joint probability density function of the state variables undergo topological changes in the probability space. While D-bifurcation analysis is quantified in terms of the sign changes in the largest Lyapunov exponent, P-bifurcation analysis is usually qualitative and through visual inspection. In this study, a new quantitative measure for P-bifurcations based on the Shannon entropy is proposed. A comparison of the parameter regimes of the noisy DVDP oscillator identified by the three bifurcation methodologies have been presented.

1. Introduction

The Duffing–Van der Pol (DVDP) oscillator is an archetypical single degree of freedom mathematical model for a range of dynamical systems, such as, aircraft wings at high angles of attack, thin panels in supersonic flows, single mode lasers with saturable absorbers and synthetic gene oscillators. The mathematical model for the DVDP oscillator is characterised by both linear and nonlinear restoring and dissipative terms and exhibits phenomenologically rich behaviour at various parameter regimes. The general form of the equations of motion are given by [1,2]

$$\ddot{X} - \alpha X - c\dot{X} + \beta_0 X^3 - \beta_1 X^2 \dot{X} + \beta_2 X^4 \dot{X} = W(t), \tag{1}$$

where $\alpha, c \in \mathbb{R}$ are respectively the linear stiffness and damping parameters, $\{\beta_i\}_{i=0}^{2} \in \mathbb{R}$ are the parameters related to the nonlinear stiffness and damping terms and $W(t)$ is the excitation. Typically, the dynamical behaviour of the DVDP oscillator is characterised by defining the parameter space in terms of the linear damping and stiffness terms. As these parameters are varied, the DVDP exhibits dramatic changes in its behaviour – from stable fixed points, limit cycles and even chaotic motion. Identification of the boundaries of the different behaviour regimes is important to gain an understanding of the dynamics associated with the system. This has led to studies devoted to the bifurcation analysis of such systems.

Bifurcations are characterised as dramatic changes in the dynamical behaviour of systems leading to topological changes in the phase space. The birth or destruction of attractors in the phase space and/or changes in the characteristics of attractors and their basins of attraction are typical examples of bifurcations. For example, a stable fixed node (an attractor) for certain parameter regimes can lead to stable limit cycle oscillations (also an attractor) as one of the parameters is changed. Typically, topological changes are quantified through invariant measures, such as the largest Lyapunov exponent (LLE), that capture the topological characteristics associated with the vector field. LLE captures the long time behaviour of two adjoining trajectories in the flow field and has been widely used in the literature to identify bifurcations.

The presence of noise in the dynamical system can lead to significant changes in its dynamical behaviour. Noise can alter the boundaries of the regimes of different attractors, the characteristics of their flow field and their basins of attraction, especially in multistable systems [3,4]. Investigations on the influence of noise on the dynamical behaviour and bifurcation characteristics have received significant attention in the literature [5–9]. However, questions related to defining stochastic bifurcations, their interpretation and tools for identifying the different dynamical regime boundaries are yet to be answered satisfactorily. Most studies in the literature define stochastic bifurcations through topological changes associated with the vector field – known as the dynamical or D-bifurcations, or through the qualitative topological changes associated with the probabilistic structure of the joint probability

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density functions (j-pdf). The latter is termed as phenomenological or P-bifurcations.

D-bifurcation analysis is based on the sudden sign change of the LLE [6–8,10,11]. The LLE, qualitatively measures the stability characteristics of a dynamical system, allows assessment of the sensitivity of the solution of a dynamical system with respect to the initial conditions, and is used for measuring the fractal dimension of strange attractors. However, the difficulties in D-bifurcation analysis lie in developing an appropriate definition for the LLE for noisy signals and devising an efficient and accurate algorithm for its computation. Algorithms for computing the LLE for noisy signals have been developed in [12,13]. The sign change of the LLE has been effectively employed as an important measure in defining the dynamical bifurcation point for a random dynamical system in the probability one sense [7,10,11]. This is mainly attributed to the fact that the LLE characterizes the asymptotic behaviour of nonlinear dynamical systems by measuring the mean exponential growth or shrinking of small perturbations to a nominal trajectory. Therefore the sample or almost sure stability of a stationary solution of a random dynamical problem depends on the LLE. However, as has been shown in [14], different algorithms could lead to different bifurcation points in the parameter space.

P-bifurcation analysis is based on the qualitative changes in the probabilistic structure of the stationary j-pdf of the state variables at different parameter regimes. The propagation of the j-pdf and the qualitative changes can be completely characterized by the solution of the associated Fokker–Planck (FP) equation [15]. The j-pdf of the state variables essentially is a measure of the time spent by a typical solution in a volume element of the phase space. Hence, P-bifurcation analysis is based on statistical information [10] and does not explicitly take into account the dynamics of the system. Thus, P-bifurcation analyses are perceived to be based on static concepts. Secondly, the j-pdf is generated by a one point motion and hence cannot be related to dynamic stability. More importantly, unlike dynamical bifurcations, changes in the topological structure of the j-pdf are not abrupt and dramatic, but occur gradually as the bifurcation parameter is changed. The boundaries of the different regimes associated with P-bifurcations are therefore not sharp.

A major source of concern in the use of D- or P-bifurcation definitions is the lack of relationship between these two approaches, often leading to identification of different stability boundaries. Studies have been shown in the literature where there have been no topological changes in the structure of the stationary pdf of the response over parameter ranges where a sign change of the LLE has been observed [16]. Similarly, it has been shown in [17] that the stationary pdf of the response of a system undergoes a topological change in its structure without any corresponding sign change of the LLE. These examples clearly show the lack of relationship between the invariant measures - j-pdf and the LLE. Questions on investigating alternative suitable invariant measures for predicting stochastic bifurcation have been raised in [7]. It was suggested in [1] that a stochastic attractor may be taken as invariant and their shape, size and their stability characteristics can be considered as essential properties whose radical changes could indicate stochastic bifurcations. This approach was used in [18] for bifurcation analysis of a stochastically excited Ueda system. Recently, qualitative changes that occur in the pdf of the amplitude of the random response has been used as indicators of stochastic bifurcation in [2,19]. The authors have used the method of stochastic averaging to obtain approximations for the pdf of the amplitude of the response processes. Stochastic averaging has been used to study a variety of low damping problems under the assumptions of quasi-harmonic regime [20]. However, stochastic averaging cannot capture the effects of the nonlinear stiffness terms as their effects get averaged out and do not enter the formulation.

The focus of this study is to investigate the bifurcation characteristics of a DVDP oscillator, in the bistable regime, subjected to white noise excitations. First, a stochastic bifurcation analysis is carried out using the existing concepts of D- and P- bifurcations, available in the literature. Subsequently, the bifurcations are examined using an invariant measure based on the Shannon entropy. Discussions on the interpretation of the dynamical behaviour of the noisy DVDP oscillator and the bifurcation characteristics are also presented. This study is expected to provide insights into the behaviour of aeroelastic systems in turbulent flows [21].

2. Stability of the deterministic DVDP oscillator

First, the stability characteristics of the deterministic DVDP oscillator is examined. Considering $W(t) = 0$, the vector field of the corresponding deterministic system is expressed as

$$\dot{X}_1 = X_2$$
$$\dot{X}_2 = \alpha X_1 + cX_2 - \beta_0 X_1^3 + \beta_1 X_1^2 X_2 - \beta_2 X_1^4 X_2. \quad (2)$$

The field is characterised by the three fixed points $(-\sqrt{\frac{\alpha}{\beta_0}}, 0)$, $(0, 0)$, $(\sqrt{\frac{\alpha}{\beta_0}}, 0)$, whose stability can be examined from the eigenvalues of the Jacobian matrix for the corresponding linearized system, given by

$$\gamma_1^{(0,0)} = \frac{c \pm \sqrt{c^2 + 4\alpha}}{2}, \quad \gamma_{2,3}^{\left(\pm\sqrt{\frac{\alpha}{\beta_0}},0\right)} = \frac{A \pm \sqrt{A^2 + 4\alpha}}{2}, \quad (3)$$

where $A = c + \alpha\beta_1/\beta_0 - \alpha^2\beta_2/\beta_0^2$. For $c < 0, \alpha < 0$ and $c^2 < -4\alpha$, $\gamma_1 \in \mathbb{C}$ with $\mathbb{R}(\gamma_1) < 0$. Hence, the fixed point at the origin is stable. For $c < 0$, as α changes from negative to positive, a pitchfork bifurcation occurs with the origin becoming unstable accompanied by the birth of two stable fixed points at $(\pm\sqrt{\alpha/\beta_0}, 0)$. The DVDP oscillator exhibits bistable behaviour in the parameter regime $-\beta_1/(8\beta_2) < c < 0$ and has two attractors – a period one attractor and a quasi-periodic attractor [2]. These attractors can be seen in the phase plane diagram shown in Fig. 1a. Here, the numerical values that have been considered are $c = -0.11$, $\beta_0 = 0.5$, $\beta_1 = 1$, $\beta_2 = 1$, $\alpha = -1$. The corresponding Poincare map is shown in Fig. 1b; the region denoted by c_1 is the basin of attraction for C_1 while c_2 denotes the basin for the attractor C_2.

A bifurcation analysis is next carried out on the basis of the computation of the LLE, with respect to the parameter α. Using the principle of Oseledec's multiplicative theorem [22], the Lyapunov exponent (LE) is mathematically defined as

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \log \frac{\| \mathbf{u}(t)\|}{\| \mathbf{u}(0)\|}, \quad (4)$$

where $\{\mathbf{u}_t: t \geq 0\}$ are the solution trajectories of the linear differential equations obtained when Eq. (2) is linearised about a reference solution $(X_1(t), X_2(t))$, for $t \geq 0$ and $\|\cdot\|$ is the Euclidean vector norm. The LLE is the maximum of the computed LEs. As the linear differential equation of $\mathbf{u}_t$ is coupled with Eq. (2), its direct solution is computationally intensive. This has led to the development of several numerical methods. In the Wolf's algorithm, the Lyapunov vectors are approximated by the set of vector obtained using the Gram–Schmidt re-orthonormalization algorithm [12,23]. An alternative algorithm proposed by Wedig [13] uses Khasminskii's unit projection theorem to compute the LLE. A negative LLE indicates stable system and a bifurcation is deemed to occur as the LLE changes sign resulting in the system becoming unstable. Fig. 2 shows the variation of the Lyapunov exponents (LE), computed using Wolf's algorithm, with α as the control parameter, for three different values of c. For $c = -0.075$ and -0.1, the LLE is observed