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# Modeling and reliability analysis of systems subject to multiple sources of degradation based on Lévy processes

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ABSTRACT

In this paper we present a framework to model the effect of multiple sources of degradation on deteriorating systems, and to find easy-to-evaluate expressions for important reliability quantities. By considering the system deterioration as a general increasing Lévy process (i.e., a subordinator), which is a process with independent, stationary and non-negative increments, the proposed methodology allows the modeling of shock-based degradation (in the form of compound Poisson processes) with different distributions of shock sizes, progressive degradation as deterministic linear drift plus a Gamma process, and multiple sources of degradation by the superposition of any of these models into the same mathematical formalism. In addition, we obtain expressions for the reliability quantities (reliability function, and probability density and moments of the lifetime $L$), and the mean and central moments of the deterioration process $X_t$. Moreover, we present efficient and accurate numerical methods to compute the reliability quantities and to simulate sample paths. Several deterioration models are compared in terms of their reliability quantities, the simulated sample paths and the feasible moments of the deterioration $X_t$. Furthermore, we propose a method to mix different Lévy models that extend the available moments with possible applications to data fitting. The results demonstrate the generality, versatility, efficiency and accuracy of the proposed Lévy degradation framework, which can open a new productive research field in the area of probabilistic degradation models.



## 1. Introduction

The evaluation and prediction of the performance of engineered systems and components over time is of particular importance, especially to support operational decisions. For instance, in electro-mechanical systems, the time-dependent assessment is needed to estimate the availability and costs of operation, inspections and maintenance actions [1]. In addition, for the particular case of infrastructure systems, it is essential to carry out life-cycle cost analysis (LCCA), to make structural reliability estimations and for directing future investments [2,3]. Evaluating the infrastructure system performance requires that we address two important aspects: the identification and modeling of the degradation mechanisms; and the evaluation of the system's lifetime distribution, from which important quantities such as the mean and moments and the reliability function can be obtained.

Let us define the system's condition at a specific point in time, $t$ as $V_t$; whereby condition we mean some mechanical properties such as the structural capacity or the stiffness. In this sense, deterioration is defined as the decay of the system condition over time; and degradation modeling is concerned with the description of the time-evolution of the deterioration. Modeling degradation requires evaluating the uncertain nature of this process [4–6]. In the literature, degradation models are usually divided into: (1) shock-based; (2) progressive; and (3) combined degradation.

Shock-based degradation models consider the cumulative effect of sudden (and independent) events (e.g., effect of earthquakes on infrastructures or over-currents in electronic devices) by removing finite amounts of the system's condition in the form of jumps or shocks at discrete points in time [7–10]. Shock-based degradation can be described by two stochastic processes: the inter-arrival times $\{T_i\}_{i\geq 1}$ (with $T_i$ the random variable (*r.v.*) of the time between the $(i-1)$th and $i$th shock) and shock sizes $\{Y_i\}_{i\geq 1}$, which represent the damage of each shock to the system. If after each shock damage accumulates and no maintenance is performed, the total deterioration by time $t$, $X_t$, can be computed as:

$$X_t = \sum_{i=1}^{N_t} Y_i, \quad (1)$$

with $N_t$ the *r.v.* of the number of shocks by time $t$. Computing the reliability for this model involves the evaluation of the distribution of the sum of $n$ random variables (i.e., shock sizes and inter-arrival

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times), which requires the evaluation of $n$-fold convolutions, and to take the limit $n \to \infty$. In general this is a difficult problem and with few exceptions Monte Carlo simulation is the only option to obtain a solution. A more tractable model assumes $T_i$ distributed exponential or Phase-type (PH), and $Y_i$ distributed exponential, normal, gamma or PH, for which simpler expressions for the convolutions are available (see, Iervolino et al. [6,11] and Riascos-Ochoa et al. [12]). The assumption of $T_i$ distributed exponential makes the process $X_t$ a Compound Poisson Process (CPP), which is easier to follow because of its Markovian property.

On the other hand, in progressive (graceful) degradation, the system's condition decays continuously over time. This is the case of processes like the wear-out of engineering devices, fatigue of materials, corrosion of metals, or deterioration of pavements [5,6]. The simplest approach to this problem is to propose a deterministic function for the deterioration $X_t$. Other models propose stochastic processes for $X_t$, like the gamma process or the Gaussian process (e.g., Brownian motion), which can reproduce the temporal variability inherent to the degradation [13]; moreover, easy-to-evaluate expressions for the reliability quantities are available. Gamma processes are more adequate because their sample paths are strictly increasing, which reproduces the monotonicity of deterioration over time (without maintenance). In fact, gamma processes have been very popular in the last years as they have been applied successfully in many cases (see van Noortwijk [4] for more details).

In many deteriorating systems both types of degradation mechanisms are present. For instance, the effect of corrosion (progressive degradation) and sudden events as earthquakes (shock-based degradation) in infrastructures is of particular importance [14]. In electronic devices sudden events like over-currents and deterioration by continuous heating of the elements are usual. It is important to propose models for these *combined degradation mechanisms*. In [5,15] a mathematical framework that includes both mechanisms is proposed in the form of a cumulative shock model (with arbitrary distributions for inter-arrival times and shock's sizes) plus a deterministic function for progressive degradation. However, the mathematical expressions obtained for the reliability quantities cannot be solved numerically in an easy way (convolutions and integrals are involved), and Monte Carlo simulations are employed. In [1] a more tractable expression for the mean of the lifetime $L$ is obtained, considering a linear deterministic model for progressive deterioration and a cumulative shock model with occurrences as a Poisson process and shock sizes distributed arbitrarily. In [6], deterioration is a combination of a gamma process and a shock-based model with occurrences following a Poisson process and shock magnitudes distributed gamma (and exponential as an special case). Easy-to-evaluate solutions were obtained for the probability of failure (and consequently, the reliability function).

However, a need still exists for proposing and solving combined degradation mechanisms that include other cases different from the previously presented. It is then necessary a flexible mathematical framework that allows the mixture of various shock processes (i.e., with several shock size distributions), different progressive degradation models (e.g., deterministic function and gamma processes), and the combination of any of these models to account for combined degradation mechanisms. More importantly, this framework should provide easy-to-evaluate expressions for the reliability quantities that solves the numerical issues presented in their computation (e.g., convolutions and infinite integrals), bringing an alternative different from Monte Carlo simulations.

In this paper we propose a novel framework to model the effect of *multiple sources of degradation* by supposing deterioration as an increasing Lévy process (subordinator), which is a process with stationary, non-negative, and independent increments. In fact, any shock model of the type CPP, and progressive models in the form of stationary gamma processes or linear deterministic models are subclasses of Lévy processes. Moreover, as the sum of any Lévy process is also Lévy, it is possible to combine any of these processes to model combined degradation mechanisms and to mix several CPP's with different distributions of shock sizes with no additional difficulty. In this sense, not only the cases reported in the literature can be reproduced with this formalism (e.g., cumulative shock CPP plus linear drift [1]; cumulative shock CPP with shock sizes distributed Gamma plus a stationary gamma process [6]), but also we can model any arbitrary combination of Lévy processes like the stationary gamma process with any CPP shock model or linear deterministic drift with gamma process.

The degradation model based on Lévy processes is used to find analytic expressions for the mean and $n$-central moments of the deterioration $X_t$ over time; as well as easy-to-evaluate formulas for the reliability function, the probability density of the lifetime $L$ and its mean (mean time to failure) and moments. Although there are some previous works that have used Lévy processes in deterioration (e.g., [16–18]), this paper is the first that explicitly address the problem of combined degradation, provides a clear definition of each mechanism based on the Lévy measure, and propose efficient and accurate numerical methods to evaluate the reliability quantities and to simulate sample paths. In addition, this paper presents a method to mix different Lévy models in order to obtain greater flexibility in reproducing deterioration moments, as an approach to data fitting.

This paper is organized as follows: Section 2 presents the problem formulation and the definition of the degrading system and the reliability quantities into consideration. In Section 3 we explain the basic notions of Lévy processes: the Lévy measure, the Lévy–Khintchine formula, and the characteristic function and exponent of a Lévy process. We also deduce expressions for the moments of a Lévy process in function of time. Section 4 is devoted to the application of Lévy processes to degradation: we explain the assumptions of the model, the special class of subordinators and the mathematical formulation that we propose for shock-based, progressive and combined degradation. In Section 5 we illustrate the formulation with several examples of Lévy degradation processes, and compare them in terms of their deterioration moments; and also we present the method of mixed Lévy processes to extend the region of feasible moments. Section 6 is centered in the reliability estimation of Lévy deterioration processes, by applying some inversion formulas to obtain their probability distribution and reliability quantities. We also propose a numerical method to solve these expressions and to simulate Lévy deterioration sample paths. These results are evaluated and illustrated with several examples in Section 7. Finally, the conclusions.

## 2. Problem formulation: deteriorating system and reliability quantities

Consider a system that is placed into operation at time $t=0$ and whose condition decreases with time. As mentioned in Section 1 the system condition at time $t$ is denoted by $V_t$, modeled as a *r.v.* that takes values in the set of positive real numbers. Its value $V_0$ at $t=0$ is deterministic and represents the condition when the system is new. The condition decreases over time due to degradation, and the accumulated deterioration until time $t$ is defined by the random variable $X_t$, with $X_0 = 0$ almost surely (a.s.). We are interested in evaluating the system reliability of a system that is abandoned after first failure and under the assumption that there is no maintenance. Hence, the condition $V_t$ is related with the deterioration $X_t$ by [5]: