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Modified shallow water equations for significantly varying seabeds

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ABSTRACT

In the present study, we propose a modified version of the Nonlinear Shallow Water Equations (Saint-Venant or NSWE) for irrotational surface waves in the case when the bottom undergoes some significant variations in space and time. The model is derived from a variational principle by choosing an appropriate shallow water ansatz and imposing some constraints. Our derivation procedure does not explicitly involve any small parameter and is straightforward. The novel system is a non-dispersive non-hydrostatic extension of the classical Saint-Venant equations. A key feature of the new model is that, like the classical NSWE, it is hyperbolic and thus similar numerical methods can be used. We also propose a finite volume discretisation of the obtained hyperbolic system. Several test-cases are presented to highlight the added value of the new model. Some implications to tsunami wave modeling are also discussed.

1. Introduction

The celebrated classical nonlinear shallow water equations were derived in 1871 by de Saint-Venant [1]. Currently these equations are widely used in practice and one can find thousands of publications devoted to the applications, validations and numerical solutions of these equations [2–4].

The interaction of surface waves with mild or tough bottoms has always attracted the particular attention of researchers [5–8]. There are however few studies which attempt to include the bottom curvature effect into the classical Saint-Venant [1,9] or Savage–Hutter[1] [11,12] equations. One of the first studies in this direction is perhaps due to Dressler [13]. Much later, this research was pursued almost in the same time by Berger, Keller, Bouchut and their collaborators [14–16]. We note that all these authors used some variants of the asymptotic expansion method. Recently, the model proposed by Dressler was validated in laboratory experiments [17]. The present study is a further attempt to improve the classical Saint-Venant equations by including a better representation of the bottom shape. Dressler's model includes the bottom curvature effects, which require the computation of bottom's profile second order derivatives. For irregular shapes it can be problematic. Consequently, we try below to propose a model which requires only first spatial derivatives of the bathymetry.

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[1] The Savage–Hutter equations are usually posed on inclined planes and they are used to model various gravity driven currents, such as snow avalanches [10].

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The Saint-Venant equations are derived under the assumption of a hydrostatic pressure field, resulting in a non-dispersive system of equations. Many non-hydrostatic improved models have long been proposed, see [18–21] for reviews. These Boussinesq-like and/or mild-slope [18,19] equations are dispersive (*i.e.*, the wave speed depends on the wavelength) and involve (at least) third-order derivatives. Although these models capture more physical effects than the classical Saint-Venant shallow water equations, they have several drawbacks. First, the dispersive effects are often negligible for very long waves such as tsunamis and tide waves. Second, the higher-order derivatives introduce stiffness into the equations and thus their numerical resolution is significantly more involved and costly than for the Saint-Venant equations. Third, the Boussinesq-like equations are not hyperbolic and, unlike the Saint-Venant equations, the method of characteristics cannot be employed (unless the operators are splitted, *e.g.* [22]). Therefore, it is not surprising that various dispersive shallow water models are not systematically used in coastal modeling.

In presence of a varying bathymetry, the shallow water equations are derived under the assumption that the bottom variations are very weak. However, even for very long surface waves, significant variations of the bathymetry can play an important role in the wave propagation. These bottom slope effects can be even more important when the wave travels over many oscillations of the seabed, due to the accumulation of bottom slope influences. Therefore, even for a shallow water long waves model, it is important to take properly into account the significant bottom variations [6]. In this article, we present a modification of the Saint-Venant equations in presence of a seabed of significant variations. This model is derived from a variational principle, which is a powerful method to derive approximations that cannot be obtained from more classical asymptotic expansions.

In the theory of water waves, variational principles are generally used together with small parameter expansions. Doing so, the approximations derived are identical to the one obtained from asymptotic expansions directly used into the equations. Thus, the only advantage of a variational method is elegance and simplified derivations. However, variational methods are much more powerful than that and approximations can also be obtained without relying on asymptotic expansions. This is specially useful when no obvious small parameter can be identified in the problem at hands.

Indeed, variational methods have been more popular in Physics [23], especially in Quantum Mechanics [24] than in Fluid Mechanics where the majority of approximate model derivations use the perturbation-type techniques. The main reason for this discrepancy comes probably from the fact that in most problems of Quantum Mechanics a small parameter cannot be simply identified (roughly speaking everything scales with the Planck constant $\hbar$). Consequently, physicists had to develop alternative methods based on the guess of the solution structure, translated into the mathematical language as the so-called solution's *ansatz*. For example, a particularly good guess of the ansatz was made by Laughlin [25] for the quantum Hall effect, which was distinguished 15 years later by the Nobel Prize in Physics in 1998.

Here, we adopt the same philosophy applying it to the long water waves propagating over a seabed with significant variations. Namely, the shallow water ansatz from [26] is additionally constrained to respect the bathymetry variations in space and time. Then, applying the variational principle, we arrive naturally to some modified Saint-Venant(mSV) equations. These mSV equations, like the classical Saint-Venant equations, are hyperbolic and can be solved with similar techniques, that is an interesting feature in the prospect of integration/modification of existing operational codes. The derivation of mSV equations presented below were communicated by the same authors in a short note announcing the main results [27]. In the present study, we investigate deeper the properties of the proposed mSV system along with its solutions through analytical and numerical methods. We specially focus on some predictions of interest for ocean modeling, in particular the fact that the waves are slowed down by the seabed slope.

This article is organized as follows. After some introductory remarks, the paper begins with the derivation and discussion of some properties of the modified Saint-Venant (mSV) equations in Section 2. Then, we investigate the hyperbolic structure and present a finite volume scheme in Section 3. Several numerical results are shown in Section 4. Finally, some main conclusions and perspectives are outlined in the last Section 5.

2. Mathematical model

Consider an ideal incompressible fluid of constant density ρ. The horizontal independent variables are denoted by $\boldsymbol{x} = (x_1, x_2)$ and the upward vertical one by y. The origin of the Cartesian coordinate system is chosen such that the surface $y = 0$ corresponds to the still water level. The fluid is bounded below by the bottom at $y = -d(\boldsymbol{x}, t)$ and above by the free surface at $y = \eta(\boldsymbol{x}, t)$. Usually, we assume that the total depth $h(\boldsymbol{x}, t) \equiv d(\boldsymbol{x}, t) + \eta(\boldsymbol{x}, t)$ remains positive $h(\boldsymbol{x}, t) \geq h_0 > 0$ at all times $t \in [0, T]$. The sketch of the physical domain $\Omega \times [-d, \eta]$, $\Omega \subseteq \mathbb{R}^2$ is shown in Fig. 1.

Traditionally in water wave modeling the assumption of flow irrotationality is also adopted. Under these constitutive hypotheses, the governing equations of the classical water wave problem are [28]:

$$\nabla^2\phi + \partial_y^2\phi = 0, \qquad (\boldsymbol{x}, y) \in \Omega \times [-d, \eta], \tag{2.1}$$

$$\partial_t\eta + (\boldsymbol{\nabla}\phi)\cdot(\boldsymbol{\nabla}\eta) - \partial_y\phi = 0, \qquad y = \eta(\boldsymbol{x}, t), \tag{2.2}$$

$$\partial_t\phi + \frac{1}{2}|\boldsymbol{\nabla}\phi|^2 + \frac{1}{2}(\partial_y\phi)^2 + g\eta = 0, \qquad y = \eta(\boldsymbol{x}, t), \tag{2.3}$$

$$\partial_t d + (\boldsymbol{\nabla} d)\cdot(\boldsymbol{\nabla}\phi) + \partial_y\phi = 0, \qquad y = -d(\boldsymbol{x}, t), \tag{2.4}$$