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Exact solutions for anti-plane deformation of a cylindrically monoclinic wedge under concentrated loads

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ABSTRACT

This article studies anti-plane deformation of a cylindrically monoclinic wedge under concentrated loads. To the best of the author's knowledge, exact solutions to this type of wedge problem under concentrated loads are not available in the literature. By applying a newly defined argument to the displacements in terms of a holomorphic function in cooperation with a complex analogous Mellin transform, the exact solutions for the considered problems are obtained. The novel arrangements greatly simplify the formulation and result in concise complex shear stress equations that can be solved. With prescribed boundary settings, the closed-form solutions for Green's functions can be derived conveniently. Exact solutions are obtained for two kinds of boundary conditions. The stress fields obtained from the two cases are presented and discussed for certain combinations of anisotropic parameters. Contours of the generalized stress intensity factor, which is related to the direction of approach to the wedge apex and the material properties, are also shown. In addition, the results of a problem with distributed loads agree well with numerical solutions. The proposed method clarifies and simplifies the analysis and solution of related wedge problems. In determining the reduced orthotropic and isotropic cases to solutions under anti-plane deformation, the results are generated naturally.

1. Introduction

The region near the apex of a wedge/notch, which is the critical area of failure in solids and structures, often has a high stress gradient and even a stress singularity. Identifying the singularity and performing a stress analysis of this region is a basic task during the structural design process. Solving such problems can provide explicit solutions for examining certain geometric problems in structures and can be used as benchmarks for similar cases in numerical procedures. Accordingly, wedge problems have attracted considerable interest. Studies of such problems contribute to our understanding of possible problems under various applied conditions, including different geometries, material types, and loading/deformation types. Anti-plane deformation of wedges involves out-of-plane shear acting on the considered problem. The shear stresses and stress intensities in the vicinity of the wedge apex are important quantities to be determined. Cases of wedges with elastic material can be categorized into four general types: (1) isotropic homogeneous material [1–4], (2) isotropic non-homogeneous material [5–11], (3) non-isotropic homogeneous [12–14], and (4) non-isotropic non-homogeneous material [15,16].

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In Type (1), attention is given to the application of various geometric shapes and boundary conditions. Basic solutions for problems of this type are useful for treating the stress singularity at the wedge apex and the stress distributions of the studied domain. The problems can be extended to Type (2) by bonding other materials to single-material cases. In addition to the applied tractions, the stress solutions are related to the ratio of material properties; this dependence is beneficial to material selections for structural designs. Furthermore, the stress intensity factor (SIF) for the bonding surface can be derived from the stress solutions. Extended studies with Types (3) and (4) should consider complex intrinsic material properties. The solutions for these problems are often sophisticated and cannot be solved analytically. However, these solutions can be directly reduced to those of isotropic cases if the boundaries are identical. In addition, the stress states can be controlled precisely when these solutions are applied to construct wedge-like structures. The stress singularity of an anisotropic wedge can be eliminated using a well-designed material, which is also known as a type of functional material.

This study considers a Type (3) material (i.e., anisotropic homogeneous), but the anisotropic material properties are based on a cylindrical coordinate system, which indicates that the material characteristics are distributed cylindrically, such as in a tree trunk and fiber-wound cylinder. As far as the author is aware of, exact solutions to this type of wedge problem of a cylindrically monoclinic material under anti-plane concentrated loads are not available in the literature. However, Ting [12] discussed the stress singularity of a cylindrically anisotropic elastic wedge under anti-plane deformation and provided the general solution expressions for the wedge. Expressions for the anti-plane deformation of a cylindrically monoclinic wedge with the degenerated solutions for the orthotropic elastic wedge have also been obtained according to prescribed boundary conditions.

As counterparts to wedge problems, Yang and Yuan [17] investigated an interfacial curved crack between cylindrically anisotropic materials under anti-plane shear. They obtained stress solutions with SIFs for the studied cases and compared the results with the cases of a planar crack. They also provided a solution for a centered planar interfacial crack in a rectilinearly anisotropic body. Li et al. [18] subsequently extended the study to a multi-crack problem and presented expressions for the stress singularities around the crack tips, image forces and torques acting on the dislocation or the dipole center. Watanabe and Payton [19] derived Green's functions for SH waves in a cylindrically monoclinic material. The wave front shape and singularity were discussed, and the contours of the displacement amplitude for the time-harmonic wave were shown.

In addition to two orthotropic elastic parameters, an additional elastic parameter is coupled to Hooke's law for a cylindrically monoclinic elastic material, which results in analytical difficulty. Nevertheless, once the Green's function solutions for the monoclinic elastic wedge problems have been obtained, the results can easily be transformed into those for general cases and can be reduced into both orthotropic and isotropic problems. This study proposes a novel analogous Mellin transform to solve the problem, which enables the Green's functions for the considered problems to be obtained conveniently. First, the novel arrangements with the newly defined argument greatly simplify the formulation and result in concise complex shear stress equations in terms of a holomorphic function that is much clearer and easier to apply to the following analogous Mellin transform. Anti-plane displacements and shear stresses are then accounted for through the considered boundary settings. This study shows that it is expedient to treat the related wedge problems of monoclinic materials under anti-plane deformation in cylindrical coordinates. Moreover, the generalized SIF is derived from the obtained stress solutions. The stress distributions on wedges and the contours of the generalized SIF for certain material property combinations are also described. Finally, the obtained stresses in a wedge problem under distributed loads are compared with those of the numerical results and show good agreement.

The anti-plane problem can be considered a model case. However, theoretical analyses are still necessary and are widely applied in many important and leading studies. For such wedge problems, they provide the correct form of the singularities and explicit solutions that can be used as benchmarks for subsequent numerical/approximate analyses, and they may be needed to improve the accuracy of numerical/approximate solutions because in practical applications, the geometry of the medium is rarely simple, and realistic materials seldom lead to analytically tractable formulations [20]. Because of the numerical treatments that are usually utilized in the in-plane formulation, the comparative analytical counterpart anti-plane problems can be used as reference cases for the model/approximation validations that precede pure numerical simulations.

2. Problem formulation and solution

Consider a monoclinic elastic material that is deformed under anti-plane shear. The stresses may be expressed in terms of the strains by Nye [21]

$$\begin{bmatrix}\sigma_{23}\\ \sigma_{13}\end{bmatrix}=\begin{bmatrix}C_{44} & C_{45}\\ C_{45} & C_{55}\end{bmatrix}\begin{bmatrix}2\varepsilon_{23}\\ 2\varepsilon_{13}\end{bmatrix}, \tag{1}$$

where σ and ε are the stress and strain, respectively, and C_{44}, C_{45}, and C_{55} are the elastic stiffness constants.

According to Eq. (1), the stress–strain relationships for anti-plane deformation in the cylindrical coordinate system ($1 \to r$, $2 \to \theta$, $3 \to z$) are (Fig. 1)

$$\tau_{\theta z} = C_{44}\gamma_{\theta z} + C_{45}\gamma_{rz}, \tag{2}$$

$$\tau_{rz} = C_{45}\gamma_{\theta z} + C_{55}\gamma_{rz}, \tag{3}$$

where τ and γ represent the shear stress and strain components, respectively.